J. R. MORELL'S

ESSENTIALS OF GEOMETRY

AS TAUGHT IN

FRANCE & GERMANY

THE

ESSENTIALS OF GEOMETRY,

PLANE AND SOLID,

AS TAUGHT IN FRENCH AND GERMAN SCHOOLS

With Shorter Demonstrations than in Euclid ;

ADAPTED FOR STUDENTS PREPARING FOR EXAMINATION,
CADETS IN MILITARY AND NAVAL SCHOOLS,
TECHNICAL CLASSES, ETC.

By J. R. MORELL,

FORMERLY ONE OF HER MAJESTY'S INSPECTORS OF SCHOOLS.

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PREFACE.

A Resolution recently passed at a Meeting of some of the principal Head Masters of schools in England has brought into additional prominence a question already strongly urged upon the attention of the nation.

The Report of the Schools' Inquiry Commission had already raised doubts as to the propriety of continuing to employ Euclid as the one authorized text-book for geometry in our public schools and Universities. The Report of the French Imperial Commissioners, published at Paris, July 1870, placed in a still more prominent light the objectionable sides of our geometrical methods [1], and the general conviction produced by these unfavourable verdicts has offered its last expression in the resolution of the meeting to which we have alluded, which urges that the Universities and teaching bodies in England be pressed to give greater freedom in the use of geometrical text-books.

Independently of the existing agitation expressing dissatisfaction with present methods in geometry, and the propriety of adopting shorter and more practical methods, a little acquaintance with the sentiments of some of our principal geometricians even twenty or thirty years ago discloses the fact that Euclid had come to be viewed by many of them as a cumbrous text-book, forced into their hands for use by the rigid, traditional practice of the colleges in which they taught.

Dr. James Thomson, Professor of Mathematics in the

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[1] Note A at the end.

University of Glasgow, in the Preface to the second edition of "Euclid's Elements" (Edinburgh, 1837), remarks,—

"In many instances I have endeavoured to improve the work, sometimes by abbreviations in the process of reasoning, and more frequently by the omission of needless repetitions in the language of the demonstrations. The notes and illustrations at the end of the volume, and those interspersed through the work, will point out some of the more important of the modifications above referred to. In addition to these it may be here mentioned, that considerable changes, and it is hoped improvements, have been made in the demonstrations of the 5th and 35th propositions of the First Book; in the 13th of the Second Book; in the 14th, 15th, 21st, 31st and 32nd of the Third Book; and in several in the Fourth, Sixth, Eleventh and Twelfth Books."

Again, at page 366 (Notes), in a footnote, Dr. Thomson remarks, "The study of Euclid would be rendered more easy to the beginner, if, in accordance with some of the preceding Notes, he should omit in the first reading the 44th and 45th propositions of the First Book, and all in the Second Book after the first three; then omitting in like manner the 35th, 36th, and 37th of the Third Book, and perhaps all the Fourth Book, he may proceed to the Fifth and Sixth Books; and having advanced in the latter to the 17th proposition, he may then turn back, and read what he has omitted, availing himself of the facilities that will be afforded, in several instances, by means of the Sixth Book. He might also postpone the 47th of the First Book, but Euclid's proof of it is so elegant and easy as to render this unnecessary; and it is desirable that he should be early acquainted with so important a proposition."

The above remarks are chiefly important because they show that in the estimation of Dr. Thomson the method of Euclid is inconsequent and therefore faulty, and that the demonstrations in many cases demand improvement, erring on the score of verbiage and repetition.

We add some observations of the same author in his Supplement to Book V., where he remarks that "The theory of proportion has now been delivered in the accurate but prolix method given by Euclid. It may suit the views and convenience . . . of many teachers and learners to have its principal properties established in a more concise and condensed form, as in the following Supplement, *so as to save time which may be more profitably devoted to other researches.*"

Even Mr. Todhunter, who would greatly regret to see Euclid discarded as the sole text-book of our public schools, admits that "it cannot be denied that defects and difficulties occur in the "Elements of Euclid," and that these become more obvious as we examine the work more closely[2]."

As regards judgments formed of Euclid by authorities on the Continent, the writer might refer to personal conversation with the French Imperial Commissioners, when one of them informed him that in the French colleges they would not think of retaining as their text-book an ancient work, full of needless repetitions and unnecessary details of proof, consuming unprofitably much time of the student, and departing widely from the clear, elegant style of proof long since admitted in France.

Without introducing larger extracts from Mr. Arnold's Report on Education in France, we may add two passages which go far to show the estimation in which our geometrical methods are held on the Continent of Europe.

The unpractical character of our mathematical teaching was frequently criticized by the eminent men with whom Mr. Arnold spoke in France. Thus he says, "They severely criticize the Cambridge teaching for devoting itself so exclusively to pure mathematics, and making the instrument into an end. The barrenness in great men and great results

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[2] Preface, p. viii., "The Elements of Euclid," by I. Todhunter. For the Use of Schools.

which has since Newton's time attended the Cambridge mathematical teaching is mainly due, they say, to this false tendency." A little farther on Mr. Arnold adds, "In general, the respect professed in France for the mathematical and scientific teaching of our secondary schools is as low as that professed for our classical teaching is high."

In another place we find this observation, with which we conclude our extracts: "I must not forget to add that our geometry teaching was in foreign eyes sufficiently condemned, when it was said that we still used Euclid. I am bound to say that the Germans and the Swiss entirely agree with the French on this point. Euclid, they all said, was quite out of date, and was a thoroughly unfit text-book to teach geometry from. . . . I was, of course, astonished, and when I asked why Euclid was an unfit text-book to teach geometry from, I was told that Euclid's propositions were drawn out with a view of meeting all possible cavils, and not with a view of developing geometrical ideas in the most lucid and natural manner. . . . At any rate, the foreign consensus against the use of Euclid is something striking, and I could not but call the Commissioner's attention to it[3]."

It is therefore evident that by a large number of competent judges, English and foreign, Euclid is regarded as an inappropriate text-book, and that a change is desired.

We proceed to offer a few additional arguments in favour of a more concise and practical text-book in geometry, and we may notice—

1st. That apart from any other reasons, the importance of saving time in this day, when the field of knowledge is so large, is not to be overlooked.

The experimental and the natural sciences have in latter years risen to such prominence that they cannot be neglected in any proper curriculum, and the recent Reports of Commissions on our Higher Education show how strongly the minds

[3] Schools' Inquiry Commission, 1868, vol. vi., p. 505-7.

of the Commissioners were impressed with the importance of giving a larger share of time and attention to them.

Accordingly, it is of the first importance to save time in education, when this can be done without any material disadvantage to each special study and to the general development of the faculties; and in the case of the shorter kinds of demonstration, in use in all the best French and German schools, it may be remarked, that they tend to produce clearness and precision, which are in some degree impeded by the wordiness and repetition of Euclid[4].

2ndly. It is admitted that the art of war, in this age, makes continually greater demands on the scientific training of officers.

Not only the engineer and artillery branches require rigid and accurate scientific training, but it is felt that the whole body of officers require a due preparation in those departments of science that have a direct bearing on their profession. It is needless to dwell on the national importance of this training, after witnessing the unparalleled success of the scientifically trained officers of the Prussian army. But it is equally evident that the excellent school of logic found in the wordy proofs of Euclid is here quite out of place. The great body of military officers require short, lucid, rapid demonstrations and practical application of theorems and problems. And here the kinds of proof and general methods in vogue in Germany and France are far better adapted than ours to secure the end in view, especially as they have furnished a number of excellent manuals and digests purposely intended to initiate students rapidly in the more essential parts of mathematical science, and particularly of geometry.

It is an attempt to produce a manual or memento of this kind that is now offered, not only for the use of military and

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[4] On "The Unsuitableness of Euclid as a Text-book of Geometry," by the Rev. J. Jones, Head Master of King William's College, Isle of Man, p. 31.

naval cadets, but also to aid all students in geometry to prepare for examination, more particularly in civil engineering; and for those members of technical classes who are devoting themselves to trades in which mensuration in all its elements enters as an important factor.

An inspection of the methods employed will show that French and German geometricians, from whom our proofs are taken, do not at one moment bind the hands of the student and at the next expect him to describe circles and other figures requiring, if they be accurate, elaborate constructions with ruler and compasses[5]. It will also appear that our Continental neighbours do not condemn the student of elementary geometry to keep a geometrical figure rigidly in the place in which it is laid down on paper. Revolution and superposition[6]—essential in the case of descriptive geometry—and other similar devices are allowed in elementary geometry abroad, simplifying and shortening the proofs, and bringing the result in many cases more evidently to the mind.

It is manifest that these mechanical and manual aids are not to be too freely used or applied in all cases, but it is also clear that to exclude them altogether arbitrarily from our demonstrations is to resort to an unnatural restriction, and to involve ourselves in contradiction, because many of our intellectual demonstrations are only possible and derived after practical and experimental testing of the axioms on which they are based.

An instance of what we mean will suffice. Euclid's eighth Axiom sets forth that magnitudes which exactly coincide with one another are equal. A moment's reflection will show that this axiom is obtained from a practical acquaint-

[5] On "The Unsuitableness of Euclid as a Text-book of Geometry," by the Rev J. Jones, Principal of King William's College, Isle of Man, p. 12.

[6] On "The Unsuitableness of Euclid," by the Rev. J. Jones, pp. 13, 14.

ance with the fact assumed, and this implies measurement and transposition.

Again, Dr. Thomson, at page 6 of his "Euclid" (Note), remarks, "These postulates require in substance that the simplest cases of drawing straight lines and describing circles be admitted to be known, and they imply the use of the rule and compasses, or something equivalent."

Here, then, it is admitted that our teaching of geometry by Euclid is based on an inconsistency, because it gives us at one moment what it withholds at another, i. e., the power of more rapid determination of results by a freer and more natural system of investigation, combining the hand, which is the source of our ideas of dimension and extension, with the intellect, which draws conclusions from the combined data of intuition and of the senses[7].

What has been said here will suffice to show that some reform in our methods of geometry is demanded by the spirit of the age and by common sense.

In the following pages only a little sample has been attempted of what may be done in a new direction, but it is trusted that it is sufficient to show that many advantages will accompany the adoption of the improved methods.

Others have preceded us in giving improved demonstrations, being innovations on Euclid, but they are confined to plain geometry, and considerably more lengthy than the present little volume.

In conclusion we may add that in dropping the distinction between theorems and problems we for once follow the example of Euclid in his original form, and that the demonstrations interspersed through the work, when they differ from Euclid's by shortening and simplifying the proof, are taken from approved authorities, chiefly French, and especially from the text-books most used in the principal Paris Lycées.

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7 On "The Unsuitableness," &c., op. cit., p. 14.

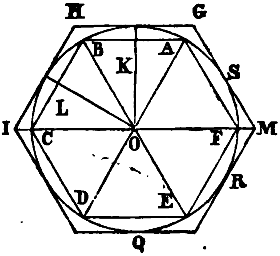
NOTE A.—"Si dans notre premier volume, nous nous étions montrés peu satisfaits de l'enseignement des mathématiques dans les écoles secondaires de l'Angleterre, nous ne sommes pas sûrs d'avoir tout à louer à Cambridge, centre officiel, pour ainsi dire, des sciences exactes. . . . Nous avons dit aussi que la géométrie est alourdie par d'interminables rédites, et qu'en voulant faire de la précision on fait du verbiage.—Qu'on nous montre un Euclide des écoles où, par exemple, à parité de format, les propositions XIV. et XV. du troisième livre n'occupent, au lieu de deux pages, que la moitié de cet espace, comme dans Legendre aux propositions II. et VIII. du deuxième livre, et nous nous empresserons de reconnaître nos torts." De l'Enseignement Supérieur en Angleterre et en Écosse, Rapport à M. le Ministre de l'Instruction Publique par MM. J. Demogeot et H. Montucci. Paris, 1870.

ERRATA.

Page 56, Fig. 72, add to top of the larger diagram B

— 58, line 15, *for* $\frac{AB}{AB'} = \frac{BC}{DC'}$ *read* $\frac{AB}{AB'} = \frac{BC}{B'C'}$

— 64, Fig. 83 (b), insert letters K and L, as in following diagram :—



Page 84, line 12, *for* intersections *read* intersection

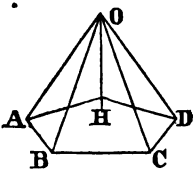
— 91, case 4, line 3, *for* equal plane, PA is perpendicular *read* equal, plane PA is perpendicular

— 92, line 2, *for* AC, parallel to AB *read* AC, parallel to BD

— 94, lines 2 and 3, *for* eight trihedral angles *read* six dihedral angles

— 95, line 14, *for* $AOB - BOC + COA$ *read* $AOB < BOC + COA$

— 98, Fig. 106, the céntre perpendicular line O H should be lengthened :—



Page 103, Fig. 111, over the bottom diagram, *for* Octahedron *read* Polyhedron

— 114, line 2, *for* prism *read* pyramid

— 122, last line, *for* $CBDF$ *read* $CBDE$

— 123, line 1, *for* and also the lines GB, GD *read* and also the line GE

— 124, line 8, *for* $EH : ED :: DG : DF$ *read* $DH : DE :: DG : DF$

THE

ESSENTIALS OF GEOMETRY.

PRELIMINARY NOTIONS.

§ I.

Object of Geometry. Extension of a body; its length, breadth, and depth. Superficies. Mathematical Points and Lines. Straight and Curved Lines; their most remarkable properties. Plane and Curved Surfaces.

1. Geometry has for its object the study and measure of extension. Extension is the space occupied by a body. It has three dimensions, which are—longitude or length, latitude or breadth, and height or depth.

The term superficies or surface is given to the sides or limits that separate a body from surrounding space in which all bodies are situated.

Surfaces have only length and breadth.

The limits of surfaces are called lines. The mathematical point has no extension.

The form of any thing extended is called its figure.

2. Lines are divided into straight and curved. A straight line is that which has all its points in the same direction, like the edge of a ruler or a string tightly stretched, &c.

Every straight line may be considered to be indefinitely prolonged.

Two points determine the position of a straight line.

If two or more straight lines have their extremities common, they coincide in their extension, and are equal[1].

The shortest distance between two points, is the straight line that unites them.

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[1] Two figures are equal when being superposed, or placed one on the other, they coincide perfectly throughout their extension.

B

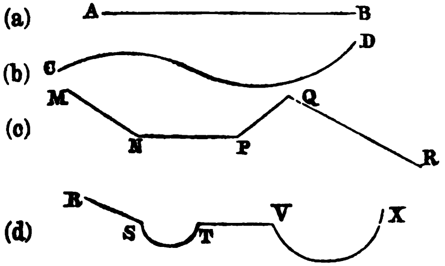
The intersection of two straight lines is the point [2].

To point out a straight line, it is usual to read the letters at its extremities, thus we say the straight line *AB* (Fig. 1, a).

A curved line is one whose points change direction, continually, such as *CD* (Fig. 1, b).

From one point to another an infinite number of lines may be drawn.

Fig. 1.



The name of broken line is given to a continuation of straight lines that do not form a single straight line, such as *MNPQR* (Fig. 1, c).

The name of mixed line is given to a continuation of straight and of curved lines; such as the line *RSTVX* (Fig. 1, d)[3].

3. Surfaces are divided into *plane* and *curved.*

A plane surface is that with which a straight line coincides if touching it in two points[4].

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[2] The term intersection of two lines is given to the point where they cut each other. See Legendre, Éléments de Géométrie, par Blanchet, 11ième édition, iv. livre premier.

[3] Every projectile describes in the air a curve called trajectory (b).

Lightning, in its course to the earth, follows a broken line (c).

Common roads, but more especially railroads, are an application of mixed lines (d).

[4] Manuel du Baccalauréat ès Lettres. Redigé par E. Lefranc, Ancien Professeur agrégé au Collège Rollin, et G. Jeannin, Ancien Professeur de l'Académie de Paris, p. 130 (1866-67).

A plane surface is also called a *plane.* Every plane may be considered as indefinitely prolonged.

Two straight lines that cut each other, or three points that are not in a straight line, determine the position of a plane [5].

The intersection of two planes is a straight line.

A *curved surface* is that with which a right line does not coincide if applied to any of its points.

A curved surface admits of a variety of species.

A continuation of planes that do not form a single plane is called a broken plane, like the surface of the staircase of a house.

A continuation of plane and curved surfaces is named a mixed surface.

§ II.

Circumference. Centre. Radii. Diameters. Chords. Tangents and Secants of the Circumference. Concentric and Excentric Circumferences. Equal Circumferences. Division of Geometry.

4. A circumference is a closed and plane curve, whereof the points are equally distant from one in the middle named the centre.

Radii are the straight lines that join the centre to the circumference. All the radii of the same circumference are equal.

Diameters are the straight lines which, passing through the centre, terminate at the circumference.

Every diameter is composed of the sum of two radii, and consequently the diameters of the same circumference are all equal.

Any diameter whatsoever divides a circumference into two equal parts; therefore, if on superposing its two halves, they did not agree, the radii of the same circumference would be unequal, which is absurd.

An arc is any portion of the circumference [6].

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[5] Éléments de Géométrie, par A. Amiot (1866), 11ième édition, 1ière et 2ième Leçons, théorème i., corollaire ii. p. 191. Figures dans l'Espace.

[6] Éléments de Géométrie, par Amiot, op. cit. 11ième Leçon (Figures Planes), Définitions, p. 40.

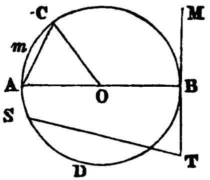
A chord is the straight line that unites the extremities of an arc.

Even though two arcs forming the circumference correspond to every chord, the smaller arc is always referred to.

A secant is the straight line that cuts the circumference.

A tangent is the straight line which indefinitely prolonged, has only one point common with the circumference. This point is called tangential, or the point of contact[7].

Fig. 2.



O is the centre of the circumference. *OA, OB, OC,* &c. are radii. (Fig. 2.)

AB is a diameter.

AC is the chord of the arcs *A m C* and *ADBC.*

TS is a secant.

And *TM* is a tangent whose point of contact is *B.*

5. The circumferences that have one and the same centre are called concentric; and those that have a different centre are called excentric[8].

In order that two or more circumferences traced in the same plane may coincide it is necessary that the centre be common, and the radii equal.

6. Elementary Geometry is divided into Plane Geometry and Solid Geometry, or, more accurately, Geometry in Space.

Plane Geometry treats of extension or extended bodies that have all their points in the same plane. Geometry in Space treats of extended bodies whose points are not in one and the same plane.

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[7] Legendre, Éléments de Géométrie, avec additions et modifications par M. A. Blanchet, Ancien Élève de l'École Polytechnique, &c. 11ième édition, livre ii., p. 31 (1862).

[8] A stone falling into the depths of a pond, or any quiet sheet of water, forms a series of concentric circumferences.

PART I.

PLANE GEOMETRY.

(A.) OF LINES.

§ III.

What is an angle, and what are the names of its elements? The magnitude of an angle does not depend on the length of its sides. Bisector of an angle.

7. A plane angle is the greater or less inclination of two straight lines to a common point[1].

The straight lines that form an angle are called sides, and the common point the *vertex* or summit.

An angle is designated by three letters, placing the letter at the vertex in the middle; but when no other letter occurs it is sufficient to read the letter at the vertex or summit.

Thus in the case of Fig. 3 and 4, these features are designated as angle AOB, angle BOC, angle P and angle Q.

8. The magnitude of an angle does not depend on the length of its sides, but on their greater or less inclination or opening[2].

Fig. 3.

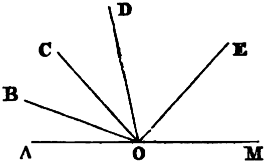
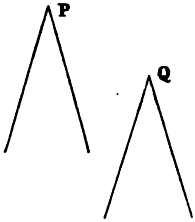


Fig. 4.



[1] Our definition accords with Euclid, Book I., n. 7 (Thompson's edition, 1837); Book I., n. 8 (Todhunter, 1867).

[2] Amiot, Éléments, &c., op. cit. p. 4—5, "La grandeur d'un angle ne dépend que de l'écartement de ses côtés," &c.

The angle AOB increases continually in proportion as the straight line OB takes the direction OC, OD, &c., and it will reach its limit, if the side OB takes the direction OM, exactly opposite to OA. In this case, as the straight lines OA and OM arè reduced to one common line, the angle no longer exists. (Fig. 3.)

9. The name of equal angles is given to those whose sides have the same opening or inclination.

It is evident that if two equal angles be superposed so that the vertex and one side coincide, the other sides must also coincide.

Reciprocally, two angles, whose sides coincide, when superposed are equal[3].

10. The bisector of an angle is a straight line which divides it into two equal angles. An angle has only one bisector. OB is the bisector of the angle AOC on the hypothesis that angles AOB and BOC are equal[4].

§ IV.

Adjacent angles, right angles, acute angles, and obtuse angles. Complementary and supplementary angles. Adjacent angles are always equal to two right angles. Angles opposed at the summit. Their equality.

11. The name adjacent angles is given to those angles that have one side common. (Fig. 5, a, c.)

Otherwise defined:—Two angles, MOP, PON, are adjacent, when they have the same summit O, a common side OP, and they are placed on the opposite sides of this common side OP[5]. (Fig. 5, a.)

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[3] Legendre, Éléments, &c., par Blanchet, op. cit. 11ième édition, p. 2 (xiii.), "Deux angles A et B sont dits égaux quand on peut les faire coïncider."

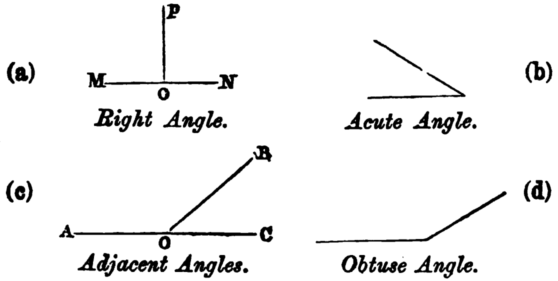
[4] Amiot, Éléments, op. cit. p. 7, problème 1, "On appelle bissectrice d'un angle, la ligne droite qui divise cet angle en deux parties égales."

[5] Éléments de Géométrie, par A. Amiot, Professeur au Lycée

AOB and *BOC* are adjacent angles, as they have one side *OB* common, while the other two sides *AO* and *OC* form one and the same straight line. (Fig. 5, c.)

If the adjacent angles are equal, each of them is called a right angle; such are *MOP* and *PON*. (Fig. 5, a.)

Fig. 5.



All right angles are equal, because they coincide when superposed [6].

An obtuse angle is an angle larger than a right angle. (d.)

An acute angle is an angle less than a right angle. (b.)

12. Complementary angles are those which, joined together, are equivalent to one right angle. *AOB* and *BOC* are complementary angles. (Fig. 6.)

St. Louis, à Paris; 1ière Leçon, No. 6, p. 3, 11ième édition (1866).

Mémento du Baccalauréat ès Lettres: Résumé des Connaissances demandées pour l'examen du Baccalauréat ès Lettres (l'Arithmétique et la Géométrie ont été redigées, par M. Bos, Professeur de mathématiques au Lycée Louis le Grand), première partie, Géométrie plane, No. 15, p. 108.

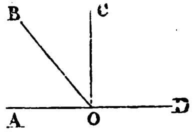
[6] Amiot, op. cit. p. 4. Corollaire—"Tous les angles droits sont égaux."

Supplementary angles are those which, when joined, are equivalent to two right angles[7]; such are AOB and BOD.

Two angles that have the same complement, or the same supplement, are equal[8].

Adjacent angles, formed by one straight line meeting another, are always supplementary; or, every straight line CD, which meets another AB, forms with it two adjacent supplementary angles[9].

Fig. 6.



13. All adjacent angles formed at one point, and on one and the same side of a straight line, are equivalent to two right angles, and the angles formed round a point are together equivalent to four right angles[1].

Fig. 7.

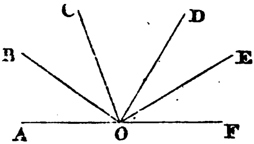


Fig. 7. For $AOB + BOC + COD + DOE + EOF = 2$ rights; and

Fig. 7. $AOB + BOC + COD + DOE + EOF = AOB + BOC + COD + DOE + EOF = 2$ rights $+ 2$ rights $= 4$ rights (doubling the figure over AF).

14. Angles opposed at the summit or vertex are angles such that the sides of the one are prolongations of the sides of the other. These angles are always equal.—Thus the

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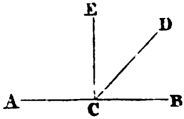
[7] Amiot, op. cit. 2ième Leçon, Définitions, p. 5.

[8] Amiot, théorème iii. p. 7

[9] Mémento, op. cit. Géométrie, No. 25. p. 110:—



At point C draw CE perpendicular to AB. This gives:

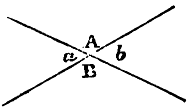
$$ACD = ACE + ECD$$
$$BCD = BCE - ECD.$$

Adding: $ACD + BCD = ACE + BCE = 2$ rights. Q.E.D.

[1] Mémento, op. cit. Nos. 26, 27, p. 110.

obtuse angles A and B are equal, because they have for supplement the same angle a. (Fig. 8.)

Fig. 8.



Similarly, the acute angles a and b are equal.

Corollary. If one of the four angles formed round the point A or B is a right angle, the others are so also. (Fig. 8.)

§ V.

Measure of angles. Division of the circumference into degrees, minutes, and seconds. Graduated semicircle or protractor. Its application to the measure of angles. Angles inscribed in a circumference. Their measure.

15 The term measure of an angle is used to signify the number of times that it contains units of the same kind.

The measure of an angle is the same as that of the arc traced from its summit or vertex, with a radius of any length and intercepted between its sides [2].

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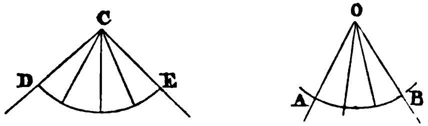
[2] If we take as unit of the angle at the centre of a circumference, the angle which intercepts between its sides the unit of arcs, the measure of an angle at the centre is the same as that of the arc contained between its sides, or, more concisely, an angle at the centre has for measure the arc comprised between its sides.

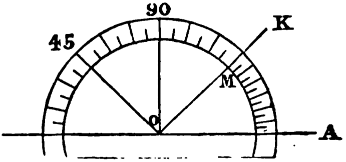
Let AOC be the angle to be measured, AOB the angular unit; from the point O, as centre, with any radius, I describe a circumference. The unit of arcs will be, by hypothesis, the arc AB intercepted between the sides of the angle of unity. (Fig. 3.) Then the measure of angle AOC will be the ratio $\frac{AOC}{AOB}$, and the measure of the arc AC will be the ratio $\frac{AC}{AB}$. But these measures are equal. For in the same circle, or in equal circles, the ratio of two angles at the centre is the same as that of their arcs. Thus, let us suppose that AB and DE have a common measure

Fig. 9 (a).



16. To estimate the value of any arc of the circumference, it is supposed to be divided into 360 equal parts, called *degrees;* each degree is divided into 60 equal parts called minutes, each minute is divided into 60 seconds, &c.

Fig. 9 (b).



Degrees, minutes, and seconds are expressed :—45° 40′ 30″.

According to this, a right angle has 90°; two adjacent angles contain 180°, and lastly all the angles formed round a point contain 360°.

17. The protractors or semicircles of mathematicians are commonly made of brass, talc, or bone, divided into degrees and half degrees.

To measure any angle whatever, by means of a protractor or graduated semicircle, it is only necessary to place the instrument so that its centre may coincide with the summit of the angle, and the diameter of the protractor with one of its sides, in which case the other side will show by the graduation of the protractor the required number of degrees and minutes for the angle.

18. The term inscribed angle is given to one that has its

---

contained 3 times in AB and 4 times in DE, then $\frac{\text{arc } AB}{\text{arc } DE} = \frac{3}{4}$.
If we then join the extremities of the arcs to their centres O and C, the angles AOB and DCE will be divided into small angles, all equal to one another. Now angle AOB contains 3 and DCE 4 of them;

Therefore $\frac{\text{angle } AOB}{\text{angle } DCE} = \frac{3}{4}$; then $\frac{\text{angle } AOB}{\text{angle } DCE} = \frac{\text{arc } AB}{\text{arc } DE}$.
(Fig. 9, a.)

summit on the circumference, and whose sides are two chords.

(Fig. 10.) Its measure is half the arc contained between its sides.

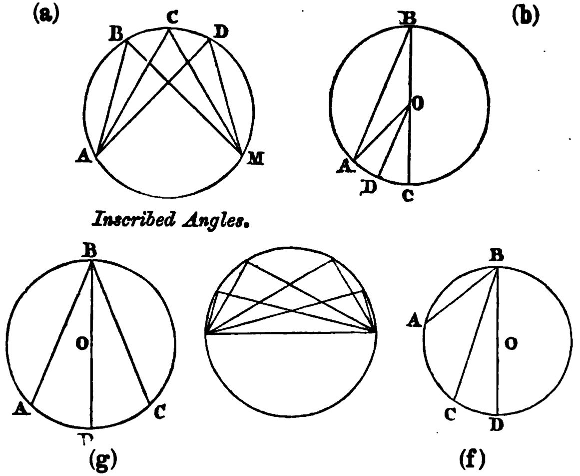
There are three cases of inscribed angles :—

1st Case. One of its sides passes through the centre (Fig. 10, b), as BC in ABC. *Construction:* Join OA, then AOB is an isosceles triangle, for $OA = OB$; therefore angle A = angle B [3]; but the exterior angle AOC = angles A and B[4]; therefore it is double of B. Now angle AOC is measured by the arc AC; therefore angle $B = \frac{AC}{2}$.

2nd Case. Centre O is inside angle ABC (Fig. 10 g). Draw the diameter BD, then angle ABC = angles $ABD + DBC$.

The measure of $ABD = \frac{\text{arc } AD}{2}$.

Fig. 10.



Inscribed Angles.

[3] § 64.

[4] For if (Fig. 10, b) at centre O we draw a parallel OD to AB, angle $COD = OBA$ (§ 25), and angle $DOA = BAO$. Therefore $AOC = OBA + BAO$. Q.E.D.

The measure of $DBC = \frac{\text{arc } DC}{2}$.

Then the measure of $ABC = \frac{\text{arc } AD}{2} + \frac{\text{arc } DC}{2} = \frac{\text{arc } AC}{2}$.

3rd Case. The centre O is outside the angle ABC (Fig 10, f). Draw the diameter BD. Then $ABC = ABD - CBD$.

The measure of $ABD = \frac{\text{arc } AD}{2}$. That of $CBD = \frac{\text{arc } CD}{2}$.

Then the measure of $ABC = \frac{\text{arc } AD}{2} - \frac{\text{arc } CD}{2} = \frac{\text{arc } AC}{2}$.

According to this, it follows that all inscribed angles, standing or resting on one and the same arc are equal[5]; all inscribed angles that stand on a diameter are right angles, for they have as measure half a semi-circumference or a quadrant, therefore they are right angles.

§ VI.

Use of the ruler and compass to construct on paper an angle equal to another given angle. Solution of the same problem. To make an angle equal to the sum of two others. To make an angle double, triple, and quadruple of another angle. To draw the bisector of an angle.

The following problems can be solved by the student on inspection, without demonstration:—

To draw a straight line through two given points.

To draw a straight line equal to another.

To find a straight line equal to the sum or difference of two other given straight lines.

To find a straight line twice, three times as large as another.

To draw an arc of a circumference when the centre and radius are given.

19. To make an angle equal to another given angle.

Let A be the given angle (Fig. 11). Describing from A with the same radius, the arc mn, and from the point B of the straight line BD, the indefinite arc pr and taking pq equal to mn, the angle B will be the required angle.

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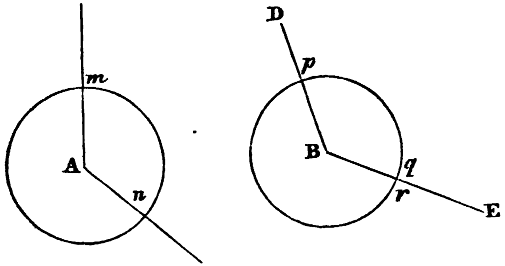
[5] Corollaries i. and ii., No. 161, p. 14 of the Géométrie Élémentaire of H. Bos, Professeur au Lycée St. Louis (1868). Euclid, III. 21.

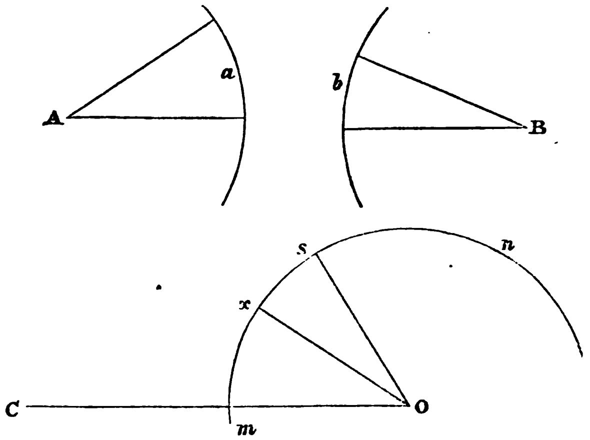
Fig. 11.



To solve the same problem by means of the graduated protractor, you measure the value of the given angle, which in the example given is 112 degrees, you place the protractor so that the centre coincides with the given point *B*, and the diameter following the line *BD*, in which case the number 112 of the graduation will give the direction of the other side *BE*.

20. *To make an angle equal to the sum of two others.* (Fig. 12.)

Fig. 12.



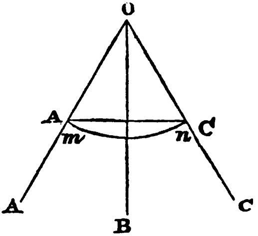
Let A and B be the given angles. Supposing O to be the summit of the required angle; you draw the straight line OC, and with the same radius you trace from the points AB and O the arcs a, b, and the indefinite one mn. Then taking on mn, the arc mx equal to a and on its continuation xs equal to b, we shall have the required angle sOC.

21. To make an angle, double, triple, &c., of another.

Drawing the arc that answers as measure for the first, you take for measures of the required angles, arcs, double, triple, &c., of the first.

22. *To divide an angle in its centre, or draw a bisector of an angle.*

Fig. 13.



Produce from O (Fig. 13) with any radius, the arc mn, and from the points m, n, with a radius greater than half their distance, draw two other arcs, whose intersections will give us the second point of the required bisector B.

§ VII.

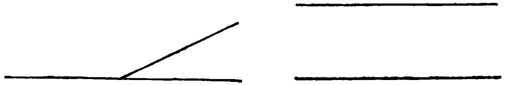
Different positions of a straight line on a plane. Perpendicular, oblique, and parallel straight lines. Angles formed by a straight line cut by two other straight lines.

23. The positions of two straight lines drawn on a plane, may generally be referred to *two* cases, in the first of which they cut one another, though they are indefinitely prolonged.

If they cut one another they can do so, forming two equal or unequal angles, from which feature the three well-known cases are deduced relating to perpendicular, oblique, and parallel straight lines. (Fig. 14.)

Perpendicular Straight Lines.

Fig. 14.



Oblique Straight Lines. *Parallel Straight Lines.*

24. Two straight lines are perpendicular to one another, when they form two or more right angles.

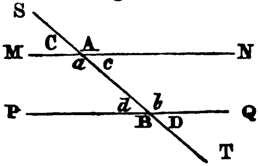
Two straight lines are oblique, if they form an obtuse or an acute angle.

Two straight lines are parallel, if when traced on the same plane they never meet, however far they may be prolonged.

It is evident that through a given point in a straight line you can draw a perpendicular and several other lines oblique to a second straight line, besides another that will be parallel to it[6].

25. If two straight lines *MN* and *PQ* are cut by a third *ST*, the latter is called the secant, and forms with the first-named straight lines (*MNPQ*) eight angles, four internal and four external. (Fig. 15.)

Fig. 15.



The internal are *a*, *b*, *c*, *d*, and the external *A*, *B*, *C*, *D*.

The name of alternate angles is given to the internal or external angles on different sides of the secant, like *a* and *b*, *c* and *d*, *A* and *B*, *C* and *D*.

The name of corresponding angles is given to those

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[6] The theory of parallel lines has given rise to much discussion. It is generally admitted that all reasonings on the subject are based on some hypothetical proposition which is assumed without demonstration. Thus it is assumed as self-evident that through a given point only one line can be drawn parallel to another. (Legendre, Éléments, par Blanchet. "On admettra," &c., prop. xxii., Livre 1, p. 21, ed. 11ième.)

See also Notes and Illustrations (Prop. xxix. p. 353) of Euclid's Elements, by Dr. Thompson, Professor of Mathematics in the University of Glasgow (1837).

situated on the same side of the secant, one being internal, and the other external.

Such are A and b, a and B, c and D, C and d.

In all these cases, these angles are supposed not to be adjacent.

§ VIII.

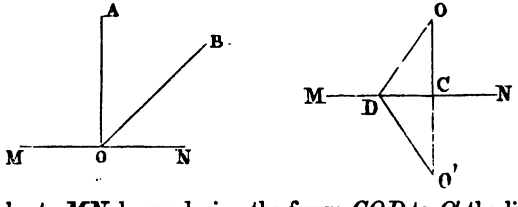
Straight lines perpendicular or oblique to one another. Distance from a point to a straight line. To draw a perpendicular to a straight line, through a given point. To divide a straight line into two equal parts.

26. A straight line is perpendicular to another, if they both form a right angle; and it is oblique to it, if they form an acute or an obtuse angle.

Through a given point only one perpendicular can be drawn to a straight line. (Fig. 16.) Thus if OA is perpendicular to MN it is absurd to suppose that OB is also perpendicular to it, because in that case, the angles MOA, MOB would be equal, which is impossible.

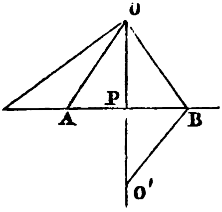
Also if OC is perpendicular to MN, OD cannot be perpendicular to it, because if OC and OD were both perpendicular to MN, by producing the figure COD to O' the lines OCO' and ODO' would be straight lines, which is impossible.

Fig. 16.



27. If from the point O, the perpendicular OP and different oblique lines are drawn, the perpendicular is the shortest of them, and the obliques OA and OB at equal distances from the perpendicular, are equal. (Fig. 17.)

Fig. 17.



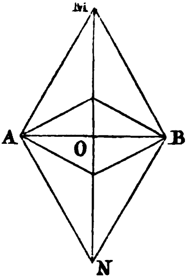
The first proposition is proved by doubling the figure over AB, in which case it results that OO' is less than OBO'; and the second proposition is proved by doubling the figure over OP, in order to verify the equality of the obliques by superposing OA and OB.

It is evident that the oblique which separates itself most widely from the perpendicular will be the greatest. It follows from this that the nearest distance from a point to a straight line is the perpendicular drawn from this point to the straight line.

From a point outside a straight line only two equal oblique lines can be drawn to it, on opposite sides of the perpendicular.

For any other line drawn from the same point O, if equal to the first OA, would coincide with the second OB, because, not coinciding with OB, it would be longer or shorter than OA; that is, unequal to it.

Fig. 18.



28. Every point of a perpendicular raised on a straight line from its centre is equidistant from its extremities.

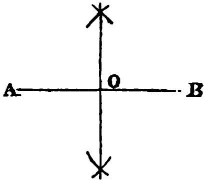
Thus each of the points of the straight line MN, is equally distant from the extremities A and B of the straight line AB.

It follows from this that if one of the points of a straight line is equidistant from two other points in the straight line on which it falls, both these straight lines are respectively perpendicular, one to the other.

29. Through a point O of a straight line to draw a per-

pendicular to it. (Fig 19.) (Euclid, Prop. xi. of the 1st Book. Problem i. Book II. of Legendre.)

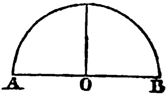
Fig. 19.



Taking with the compasses, the distance $OA = OB$, and with the centre first at A and then at B, taking a radius of any length (provided it be greater than half AB), describe two arcs, of which the intersections will give us the desired straight line.

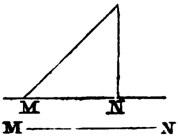
To solve the same problem with the protractor, all that is necessary, is to place it so that the centre may coincide with O, and that the diameter may follow the straight line AB, in which case, the graduation of 90° will give us the direction of the perpendicular required. (Fig. 20.)

Fig. 20.



With the square, the problem may be solved even more easily, for you have only to place the side MN against the given straight line in such a way that N may coincide with the point O. In that case, drawing a pencil along the other side of the square, we shall have the required perpendicular.

Fig. 21.



30. Through a point O outside a straight line to draw a perpendicular to it[7] (Euclid, Prop. xii. B. I. Amiot, Théorème i. Leçon vi. Eleménts). (Fig. 22.)

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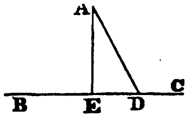
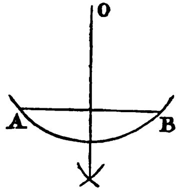
[7] The student can himself solve Problem 30, employing the protractor or square, or thus :—Join A to D in line BC. Measure angle ADB and make DAE = complement of ADB. Then AE is perpendicular to BC, AED being a right angle.

Fig. 22.



Let an arc be described from O, cutting the said straight line, and from the points of intersection A and B, as centres, and with the same or a greater radius, let two arcs be described. Then let their point of intersection be united to the given point O, and you have the required perpendicular. (Fig. 23.)

Fig. 23.



31. *To divide* a *straight* line AB, into *two equal parts*. (Fig. 24.)

Making each extremity of the line A and B a centre, and with a radius greater than half the line, draw two arcs on both sides of the given straight line. The points of intersection of these arcs being joined by the straight line mn, it will be perpendicular to AB, and O will be the required point.

Fig. 24.

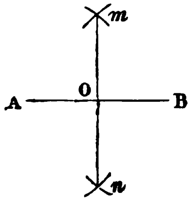
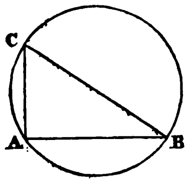


Fig. 25.



32. To draw a perpendicular at the extremity A of a straight line which is limited (Fig. 25). Describe a circumference passing through A and any other point of the given straight line. Draw from this point the diameter CB, and uniting its extremity C with A you will have the straight line AC perpendicular to AB[8].

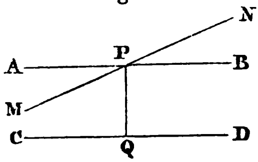
[8] Because every inscribed angle standing on a diameter is a right angle, every inscribed angle having for measure half its arc (see Fig. 10, b, g, and f, and note to § 18).

§ IX.

Parallel straight lines. Two straight lines parallel to a third are parallel to each other. Alternate and corresponding angles between parallels. Parts of parallels intercepted between other parallels. Angles that have their sides parallel.

33. One straight line is parallel to another, when both, being drawn on the same plane, never meet, though prolonged to any length.

Fig. 26.



Two straight lines perpendicular to a third are parallel to one another; because, if it were not so, from the point where they would meet we should have two perpendiculars to the same straight line, which is absurd. (Fig. 26)[9].

Thus the straight lines *AB* and *CD*, both perpendicular to the straight line *PQ*, are parallel.

34. Two straight lines, one perpendicular and the other oblique to the same straight line, are not parallel to one another.

CD being perpendicular to *PQ*, and *MN* being oblique to *PQ*, *MN* and *CD* are not parallel; consequently, if sufficiently prolonged, they would meet.

35. Through one and the same point outside a straight line only one line can be drawn parallel to the other straight line.

It is often assumed as a self-evident proposition, that through a given point only one parallel can be drawn to a straight line[1].

If through the point *P* you draw the straight line *AB*

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[9] Amiot, 7ième et 8ième Leçons, Théorème i.

[1] "On admettra, comme une proposition évidente, que par un point on ne peut mener qu'une seule parallèle à une droite." Legendre, Éléments, Livre I. Prop. xxii. p. 21, 11ième édition.

This is the opposite of Euclid's 12th Axiom, similarly assumed to base upon it his theory of parallels.

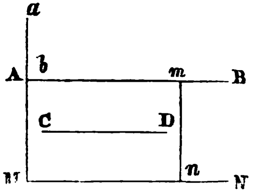
parallel to *CD*, no other line can be drawn through that point except *AB* parallel to *CD*.

Two straight lines parallel to a third are parallel to one another.

AB being parallel to *CD*, and *CD* parallel to *MN*; *AB* and *MN* will be also parallel to one another. (Fig. 27.)

For if the straight lines *AB* and *MN* met at a common point *P*, two parallels to *CD* would be drawn through the point *P*[2].

Fig. 27.



If a straight line is perpendicular to one of two parallels, it will be so to the other. If the straight line *abM* is perpendicular to *AB*, it will be also perpendicular to *MN*, supposing the latter *MN* to be parallel to *AB*. Let *AB* and *MN* be parallel. Every straight line *ab*, perpendicular to one *AB*, is perpendicular to the other *MN*. For if it were not so—that is, if *abM* were oblique to it, at the point *M*—two parallels could be drawn to *AB*, passing through it[3].

If two straight lines are parallel, their perpendiculars will be also parallel to each other.

That is, if the straight lines *AB* and *MN* are parallel, and *ab* is perpendicular to *AB*, and *mn* is perpendicular to *MN*, the straight lines *ab* and *mn* will be parallel to one another[4].

36. If two straight lines are parallel, and are cut by a secant, the alternate angles are equal, and also the corresponding angles. Conversely, if two straight lines cut by another form equal alternate or corresponding angles, the said lines will be parallel. (Fig. 28.)

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[2] Legendre, op. cit., Prop. xxiv. Liv. I., 11ième édition, p. 22. Euclid's proof, Prop. xxx. B. I., is longer.

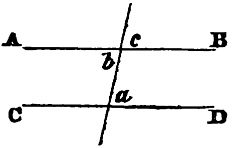
[3] Prop. xxiii. Liv. I., Legendre, op. cit. 11ième édition.

[4] In Legendre this is a corollary to Prop. xxxi. of his 1st Liv. p. 28, 11ième édition.

That is, if AB and CD are parallel, the angles a and b are equal, and also the angles a and c [5].

If a and b are equal, or if a and c are equal, the straight lines AB and CD will be parallel.

Fig. 28.



For two alternate internal, or alternate external, or corresponding angles are both either acute or obtuse, and consequently equal in the case of parallels. But if of two internal or external angles on the same side, one is acute, and the other obtuse, these angles are therefore supplementary [6].

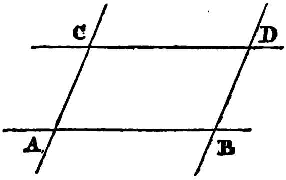
37. The parts of parallels intercepted between other parallels are equal. (Fig. 29.)

It is thus seen that AB is equal to CD, and AC to BD.

For AB and CD are two opposite sides of a parallelogram.

From this it is inferred, that if two straight lines are parallel, the perpendiculars intercepted by them are equal.

Fig. 29.



38. The angles, of which the sides are parallel, are equal or supplementary, according as both are acute or obtuse, or the one acute and the other obtuse. (Fig. 30, a.)

The angles A and B are equal, A and C also; but A and D are supplementary. (First Case.) Likewise the angles, which have sides respectively perpendicular the one to the other, are equal or supplementary. Consequently, the obtuse

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[5] This is ascertained by verifying that the adjacent external and internal angles of the same side of a secant are together equal to two right angles.

[6] Amiot, 7ième et 8ième Leçons, p. 26, 11ième édition, Théorème iv.

angles M and N are equal, but M and P are supplementary [7]. (Second Case.)

Fig. 30 (a).

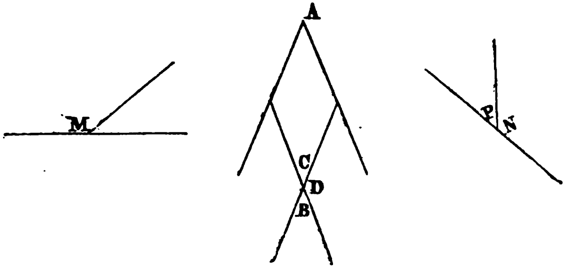
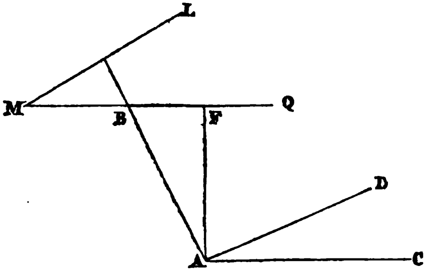


Fig. 30 (b).



First Case.

A and B are equal as corresponding angles. (36.)
A and C are equal, because C is equal to B.
A and D are supplementary, because D is supplementary to C. (36.)

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[7] Amiot, Éléments, Leçon ix., Théorème i. et ii.

Second Case.

Let DAC be $= LMQ$.

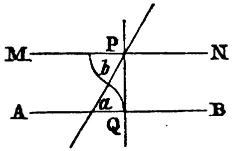
At A make AB perpendicular to AD, and AF perpendicular to AC; then AB and AF will be respectively perpendicular to ML and MQ, and in the same direction, therefore, angle $BAF = LMQ$.

But $BAF + FAD = 1$ right; and
$FAD + DAC = 1$ right. Therefore
$BAD = FAC \therefore BAF = DAC$.
(Fig. 30, b.)

39. Through a given point P outside a straight line AB, to draw a parallel to it. (Fig. 31.)

Drawing through the given point PQ a perpendicular to AB, and another MN, perpendicular to the previous perpendicular, we shall have MN parallel to AB.

Fig. 31.



To solve this problem by means of the square, adjust its longer side to the given straight line, make a ruler coincide at the same time with one of the other sides, and keeping the ruler fixed, move the square till its longer side passes through the given point.

Another construction.—Through P let a straight line be drawn, cutting the given straight line any how, and forming at the said point an angle b, equal and alternate to a; the straight line MN will be parallel to AB[8]. (Fig. 31.)

[8] Amiot, Éléments, op. cit., Problème iv. 17ième Leçon, p. 71.

§ X.

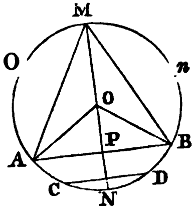
The diameter is the greatest of all chords, and divides the circumference into two equal parts. Diameters perpendicular to each other. Equal chords correspond to equal arcs, and the greater chord corresponds to the greater arc, and conversely. A diameter perpendicular to a chord. Parallel chords. Equal chords.

40. The diameter is the greatest of chords[9]. For as MN is equal to $MO+OA$ (Fig. 32) and $MO+OA$ are together greater than MA—a broken line being longer than a straight line—it is evident that MN is greater than MA.

41. Every diameter divides the circumference into two equal parts; for if on placing one half on the other they did not coincide, the radii of one and the same circumference would be different in length, which is absurd. Two diameters, perpendicular to one another, divide the circumference into four equal parts, which are called *quadrants*.

42. If two arcs of the same circumference are equal, the chords subtending them will be also equal; and if the arcs are unequal, the greater chord will correspond to the greater arc, and conversely (Fig. 32.)

Fig. 32.



Thus, if arcs MA and MB are equal, we may safely affirm that their chords will be equal; and also, if the arc ANB is greater than CND, the chord AB will be greater than CD. (Case 1, Fig. 32.)

On the other hand, if the chords AM and MB are equal, their respective arcs will be also equal; and if the chord AB is greater than CD, the arc of the former will be greater than the arc of the latter. (Case 2.)

Demonstration of the 1st Case by superposition. Folding the semicircle MBN over the diameter MN, arc MB will coincide

[9] Prop. ii. Liv. II., Éléments de Géométrie, par Legendre. Avec additions, par Blanchet, 1862.

with arc *MA*, and chord *MB* will coincide with *MA*, as point *B* falls on point *A*.

Demonstration of the 2nd Case. Let chord $MA =$ chord MB; then uniting points *A* and *B* to centre *O*, triangle $AOM =$ triangle *BOM*, and both are isosceles, having radii for sides. Therefore, folding semicircle *MBN* over the diameter *MN*, the points *MB* coincide with *MA*, and arc $MnB =$ arc *MOA*[1].

43. Every diameter perpendicular to a chord, divides it in the centre, and also the corresponding arc.

Thus *O* being at an equal distance from *A* and *B*, *M*, *P* and *N* will also be equally distant from the chord *AB*; and as equal arcs correspond to equal chords, the truth of the proposition is self-evident.

In the same way it can be verified, that if a straight line is perpendicular to a chord, and divides it in the centre, it will pass by the centre of the circumference[2].

44. Two parallel chords intercept equal arcs. *AB* and *CD* (Fig. 32) being parallel, the arcs *AC* and *BD* will be equal. For drawing the radius *ON* perpendicular to *AB*, it will be also perpendicular to its parallel *CD*, and we shall have $AN = NB$, and also $CN = ND$; hence subtracting the latter from the former, we have $AC = BD$. Q.E.D.

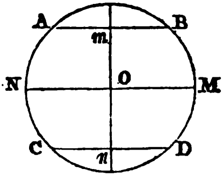
45. Equal chords are equally distant from the centre. (Fig. 33, a.)

This can be proved by doubling the figure over its diameter *MN*, drawn through the centre of the arc *AC*.

It is also equally easy to see that of unequal chords, the greater is nearer to the centre.

In fact, they cannot be equal; because equal chords are equally distant from the centre, and of unequal chords the smaller is farther from the centre.

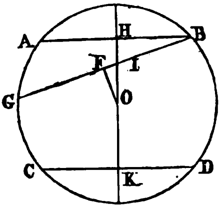
Fig. 33 (a).



[1] Amiot, op., cit., 11ième Leçon, Théorème iii. p. 41.

[2] This is treated as a corollary to the proposition that the diameter perpendicular to a chord divides it into two equal parts. Géométrie Elémentaire, par H. Bos, Professeur au Lycée St. Louis (Paris), 1868; Première année, p. 2, n. 138, 139.

Fig. 33 (b).



In circle ADC let chord BG be greater than chord CD. It is then nearer to the centre. On arc BAG, hypothetically larger than arc CD, I take a length $BA = CD$ and I draw BA. Chords BA and CD are equal, and therefore equally distant from the centre. We have then to prove that chord BG is nearer the centre than chord BA.

To prove this; drop perpendicular OF on chord BG, and perpendicular OH on chord BA. The centre H of chord BA, and centre of circle O being on opposite sides of chord BG, the straight line OH cuts BG at a point I. Now the straight line OF, perpendicular to BG, is shorter than the oblique OI: à fortiori, it is shorter than $OI + IH$, or OH. Therefore, chord BG is nearer the centre than BA.

§ XI.

Straight lines. Secants and tangents to a circumference. Different positions of circumferences. To draw a circumference through three given points. Given the point of a circumference to draw a straight line tangent to it.

46. A straight line cannot have more than two points where it meets the circumference; for if it had three such points, the straight lines drawn from them to the centre would be equal, as radii, and we know that from one point, you cannot draw three equal straight lines to another straight line[3]. (No. 27.)

47. Every straight line, perpendicular to a radius, at its exterior extremity, is tangent to the circumference; and reciprocally every straight line tangent to a circumference is perpendicular to the radius drawn to the perpendicular. (Fig. 34.)

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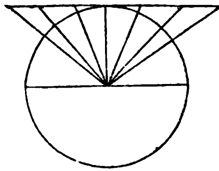
[3] Legendre, op. cit., Liv. II. Prop. iii.

Fig. 34.



For in the first case, the straight line perpendicular to the radius will have all its points outside the circumference, excepting one only; therefore it will be a tangent. And in the second case, as the radius is the shortest straight line that can be drawn from the centre of the circumference to the tangent, the radius and the tangent will be perpendicular to each other[4].

From this proposition the four following ones are easily deduced:—

CORROLLARIES.

1. Through a given point in a circumference only one straight line tangent to it can be drawn.

2. The perpendiculars drawn to the extremity of a diameter are parallel tangents.

3. The arcs intercepted by a parallel chord and tangent are equal.

4. Every straight line oblique to a radius, at its exterior extremity is a secant of the circumference.

48. The relative positions of the circumferences described on the same plane are the following. (Fig. 35)[5].

1. If they have no point in common, and are exterior to each other. In this case it can be verified that the distance of the centres exceeds the sum of the radii of both circles.

For $CC' = CA + C'A' + AA' \therefore CC' > CA + C'A'$.

2. If they have no point in common, and one is inside the other; the distance of the centres is less than the difference of the radii.

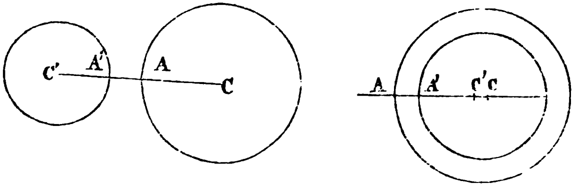
[4] Amiot, op. cit., 13ième Leçon, Théorème ii. Legendre, op. cit., Liv. II. Prop. ix.

[5] Legendre, op. cit., par Blanchet, Liv. II. Prop. xii. to xv., 11ième édition, 1862.

For $CC' = CA - C'A' - A'A$, whence $CC < CA - C'A$.

Fig. 35.

Case 1. *Case* 2.



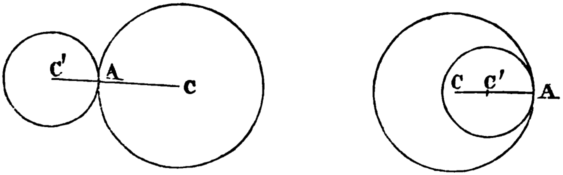
3. If they are tangent circles, or have only one point in common, being exterior to each other; the distance of the centres is in this case equal to the sum of the radii. For the point of contact A being on the line of the centres, it follows evidently that $CC = CA + AC'$.

4. If they are tangent, and one is inside the other, the distance of the centres is equal to the difference of the radii.

For the point of contact A is on the line of the centres, and this gives $CC' = CA - C'A$.

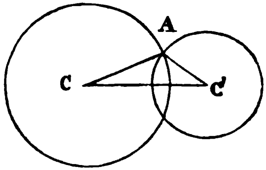
Fig. 35.

Case 3. *Case* 4.



5. If they have points in common or are secants, the distance of the centres is less than the sum of the radii, and greater than their difference.

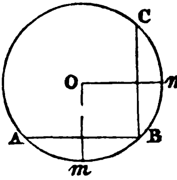
Fig. 35.—*Case 5.*



For joining the centres to one of the points of intersection *A*, a triangle will be formed, in which the line of the centres *CC'* and the radii *CA*, *C'A*, will be the three sides. Now in any angle one side is smaller than the sum of the two others, and greater than their difference. (See 69.)

49. Given three points which are not in a straight line, to describe a circumference through them.

Fig. 36.



Let *A*, *B*, and *C* be the given points. (Fig. 36.)

Joining these points by the two straight lines *AB* and *BC*, the intersection of the perpendiculars on these straight lines, and bisecting them, will be the centre of the circumference required.

The same construction is employed to find the centre of an arc, or that of a given circumference.

Demonstration. Having found *O* at the intersection of the perpendiculars; with radius *OC* describe the circle *CBA*. Then the point *O* is equally distant from *A*, *B*, and *C*; because it is on each of the perpendiculars *On*, *Om*, carried through the centre of the lines *CB*, *AB* [6].

Given the centre, the circumference can be easily described.

50. Through a given point in a circumference, to draw a straight line, tangent to the same circumference. (Fig. 34.)

Let the radius be drawn meeting the required point, and the straight line perpendicular to this radius, and that passes by the same point of contact, will be the required tangent.

[6] Amiot, op. cit., Problème v. Leçons 18 et 19, p. 76, 11ième édition.

§ XII.

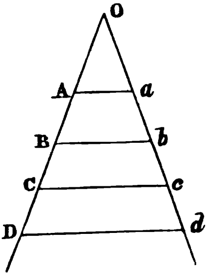
Proportional lines. Two or more parallel straight lines divide the sides of an angle into equal or proportional parts. The perpendicular to the diameter is a mean proportional between its segments. The tangent is a mean proportional between the whole secant and the exterior part of it.

51. Four lines are said to be proportional when their numerical values can form a proportion.

The numerical value of a line means the number of times that it contains the unit of length, as an inch, a foot, a yard, &c.

52. If on one side of an angle equal lengths are taken, and through these points of division lines be described parallel to each other, they will cut off equal parts on the other side of the angle. (Fig. 37.)

Fig. 37.



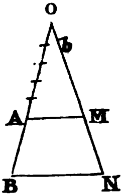
Thus taking $OA = AB = BC = CD$, and describing through the points A, B, C, D, straight lines parallel to each other, you can verify the equality oa, ab, bc, and cd (by a parallel from a to AB and on Bb).

53. If on one side of an angle, unequal parts be taken, and through the points of division parallels are described, parts proportional to the other divisions will be intercepted on the other side of the angle. Thus we shall have—

$$OA : AB :: OM : MN.$$

To prove this: suppose that OA and AB contain a common measure 5 and 3 times, and that parallels are described through the points of division; the straight lines OM and MN will also contain 5 and 3 times their common measure b; hence the proportion is evident. (Fig. 38.)

Fig. 38.

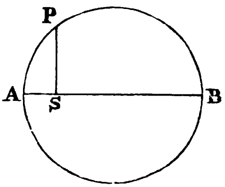


54. If from any point of the circumference you let fall a perpendicular on the diameter, this perpendicular will be a mean proportional between the segments of the diameter[7].

Thus, if PS is perpendicular to AB, you can verify the proportion:

$$AS : SP :: SP : SB.$$[8] (Fig. 39.)

Fig. 39[8].



55. If from a point outside the circumference you describe a tangent and a secant, the tangent is a mean proportional between the secant and its external segment.

If the straight line OB is a tangent to the circumference at a point B, we shall have—

$$OA : OB :: OB \,.\, OC.$$ (Fig. 40, 1.)

Fig. 40 (1).

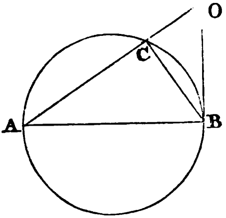
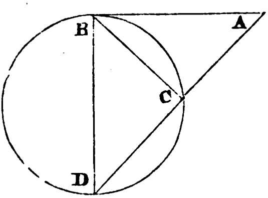


Fig. 40 (2).



This theorem can be proved immediately in the following manner: I join the point of contact B of the tangent to the points C and D where the secant cuts the circumference. The triangles ABC, ABD are similar, for they have the

---

[7] The term segments of a straight line is given to the distances from its extremities to any given point in the line.

[8] AS, SP, and SP, SB, being corresponding sides in two similar triangles.

angle A common; and their angles ABC, ADB are equal, because they are measured by the half of the same arc BC, included between their sides. It results from this that

$$\frac{AD}{AB} = \frac{AB}{AC}$$

that is to say, that the tangent AB is a mean proportional between the secant AD and its exterior part AC[9]. (Fig. 40, 2.)

§ XIII.

To divide a straight line into equal parts. To divide a straight line into parts proportional to another given straight line. To find a fourth line proportional to three given straight lines. Construction of scales of equal parts. Scale of a thousand parts.

56. To divide the straight line AB into three equal parts. (Fig. 41.)

The problem will be solved by drawing through one of the extremities of the straight line, another straight line of

Fig 41.

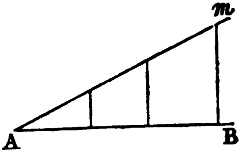
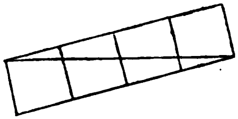


Fig. 42.



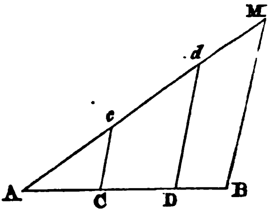
indefinite length Am, taking in this latter line 3 equal parts, beginning from A, uniting the extremities B and m, and drawing through the other points of division parallel straight lines.

The second figure (Fig. 42) indicates another construction of this same problem.

[9] Amiot, Problème i., 25ième et 26ième Leçons.

57. To divide a straight line into parts proportional to those of another given straight line. If AB is the given straight line, and AM the line that we wish to divide, let the points B and M be united by the straight line BM, and drawing through c and d other lines parallel to BM, the problem will be solved.

Fig. 43.

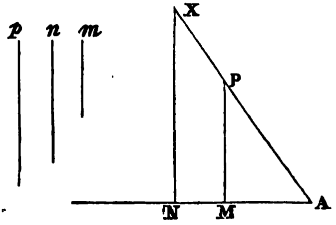


58. To find a fourth proportional to three given straight lines,

$$m : n :: p : x. \quad \text{(Fig. 44.)}$$

From the vertex of any angle, let the two first terms of the proportion be taken on one of its sides, that is, $AM = m$ and $AN = n$; let $AP = p$ be taken on the other side; and uniting the points M and P, the parallel NX will give us the straight line we seek, that is, AX.

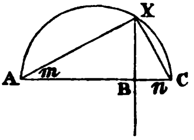
Fig. 44.



59. To find a mean proportional to two given straight lines m and n. (Fig. 44.)

On a straight line of indefinite length take the straight line $AB = m$, and in its prolongation take $BC = n$, and describing on these two lines a semi-circumference, the perpendicular BX will be the mean proportional required.

Fig. 45



Demonstration. AC = hypothenuse of a right-angled triangle AXC. But the triangles AXB and BXC are similar. (See 55.)

$$\text{Therefore } \frac{AB}{BX}=\frac{BX}{BC} \quad \text{Q.E.D.}$$

60. The scale is a straight line divided into equal parts, representing certain units of length, such as inches, feet, yards, metres, kilometres, miles, leagues, &c.

61. The mere inspection of a graduated rule is sufficient to convey a correct idea of its construction and use.

62. Topographical plans always point out the scale to which they are referred, whether it be expressed graphically at the bottom of the plan, or expressed by a fractional unit of this form $\frac{1}{500}$, $\frac{1}{1000}$, &c.

The scale $\frac{1}{500}$ means that the distances of the territory, or object represented by the plan, are 500 times greater than those on the paper; $\frac{1}{1000}$ means that the distances of the territory are 1000 times greater than those of the plan, and, consequently, that the millimetres represented on the plan express metres in nature, and that the metres of the plan are kilometres in nature, &c.

To verify the distance from one point to another represented on the plan, take this distance with the compass, and the number of parts of the rule comprised between the opening of the compass will be the number of inches, feet, metres, kilometres, and leagues, separating the said points in nature.

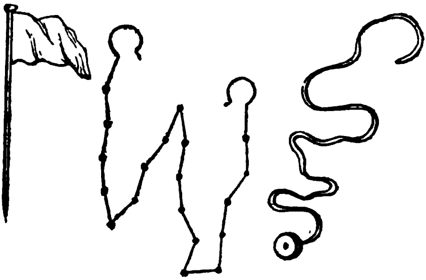
§ XIV.

Surveying rods. Chains and plumb-lines. Use of these instruments to determine a straight line in nature. To describe in nature perpendiculars and parallels to a given straight line. Measure of accessible distances.

63. Surveying rods or staffs are wooden poles provided with an iron point to fasten them into the ground, having a length of one or two yards, divided into inches and feet, or into

decimetres and centimetres, and arranged so as to carry a flag or some object (in French "mire") at the top, visible at a great distance. (Fig. 46.)

Fig. 46.



The chain consists of a hundred iron links, each a decimetre in length, in the case of French chains; it ends in two great rings, and in measuring any straight line not vertical, to compensate for the error caused by the unavoidable curvature of the stretched chain, an additional decimetre, divided into equal parts among all the links, is given to it.

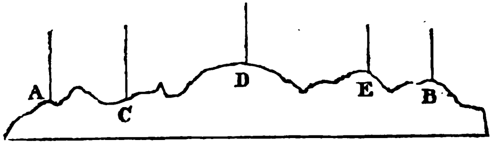
The chain is accompanied by a certain number of iron skewers or arrows of from $2\frac{1}{2}$ to 3 decimetres in length (·8225 or ·987 feet long). Instead of the chain, a tape specially prepared can be used, having a length of 20 to 25 metres, or yards.

The plumb-line is a small conical weight, sustained by a string from the centre of its base. When the other extremity of the string is suspended and the plummet remains still, the direction of the string is a vertical line.

64. It happens frequently that we have occasion to determine a straight line between two points in nature and other more distant points in the same line and direction.

For instance, let us suppose the given points to be *A* and *B*. (Fig. 47.)

Fig. 47.



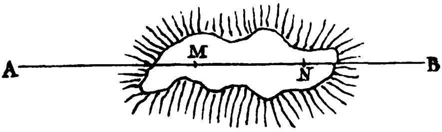
Let two surveying-rods be planted at the required points, as vertically as possible, and then let another rod be planted equally vertically at the point *C*, so that looking from *A* to *C*, *B* is covered and concealed.

Repeating the same operation at the points *D* and *E*, &c., we shall determine the straight line required.

The straight line (in nature) can be prolonged to the left of *A*, or to the right of *B*, as far as you wish.

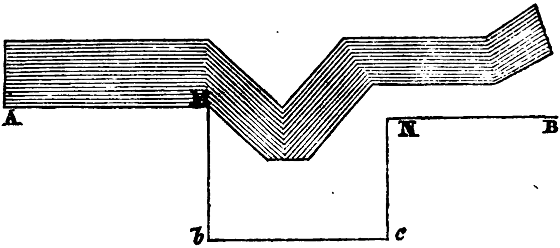
If between the given points, there is some elevated ground, it is only necessary to plant in additional rods between *A* and *B*, so that they are in a straight line with *A* and also with *B*. (Fig. 48.) Having done this (a very easy operation), the latter case is reduced to the former, and the rising ground identified with level ground.

Fig. 48.



If two intermediate rods are not sufficient, three or more can be placed. All the poles are in their true position, when each one is in the same line with the two others. If in determining the straight line you meet any obstacle, as happens in erecting new buildings in a winding, narrow

Fig. 49.

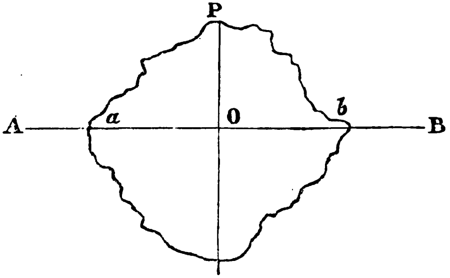


street, you raise on the straight line AM (Fig. 49), a perpendicular Mb. Then through c you draw cN perpendicular to bc, and equal to Mb, and lastly NB perpendicular to this last perpendicular cN will be the required straight line.

65. To draw a perpendicular bisector of a given straight line (in nature). (Fig. 50.)

If the given point is O, let equal distances be taken Oa and Ob, and fixing at a and b the two extremities of a cord, the middle point P,—the most distant point possible from AB,—will be the second point of the perpendicular required. P being the second point, the middle of the cord will be

Fig. 50.



fastened to it, and its extremities will give us the points a and b, equidistant from P. Then raising the cord or chain on the lower side of AB, the central point will give us the third point of the perpendicular required,—which is thus the common place, or *locus*—in French *lieu* of the points AB.

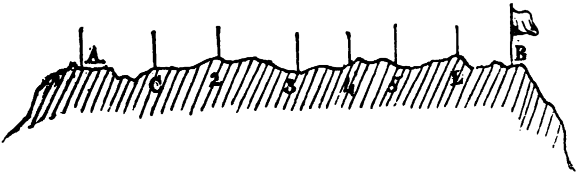
A second point of the perpendicular may be also determined by dividing the distance ab into two equal parts.

66. To draw a parallel to a straight line.

Let a perpendicular be drawn through the given point to the given straight line and through the same point another perpendicular to this perpendicular, and we shall have the required straight line.

67. To measure the distance between two given points. (Fig. 51.)

Fig. 51.



The line is determined by separating the rods from each other by distances equal to the chain or tape (which may be assumed to have 25 yards), and in this manner we shall easily determine the total distance AB; because referring to the figure (51) we shall have, for example: $AB = 6$ times $AC + LB = 150 + 10 = 160$ yards.

(B.) OF FIGURES TERMINATED BY LINES.

§ XV.

Definition of the triangle and number of its elements. The triangle may be equilateral, isosceles, scalene, right-angled, obtuse-angled, and acute-angled. The most remarkable properties verified in the case of all triangles.

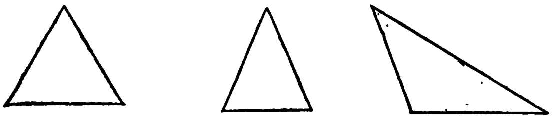
68. The name of triangle is given to a figure enclosed by three lines.

The sides of the triangle are the lines that enclose it. Angles are the inclination of the sides to each other, and vertices or summits are the intersections of the sides, (Fig. 52.)

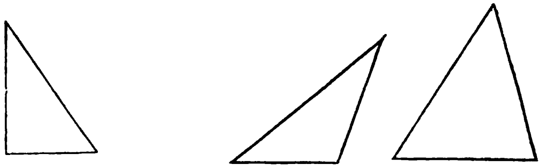
Triangles, in relation to their sides, are divided into equilateral, isosceles, and scalene.

A triangle is called *equilateral* if it has three equal sides; *isosceles*, if it has two equal sides; *scalene*, if it has three un-

Fig. 52.



Equilateral. *Isosceles.* *Scalene.*



Right-angled. *Obtuse-angled.* *Acute-angled.*

equal sides. And in relation to their angles, triangles are also divided into rectangular or right-angled, if a triangle has a right angle; obtuse-angled, if it has an obtuse angle; and acute-angled, if its three angles are acute [1].

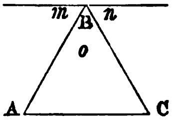
Obtuse and acute-angled triangles are also named oblique-angled triangles. The base of a triangle is any one of its sides, though the term is generally given to its lowest side.

The height is the perpendicular drawn from the vertex or summit, opposite to the base, to the same base or its prolongation.

69. In every triangle the following propositions can be verified:—

One side is less than the sum of the two other sides, and greater than their difference. (Fig. 54.) That is, the side AC, for example, is less than the sum of AB and BC, but greater than the difference between these same lines.

Fig. 54.



For AC being the shortest distance from A to C, the side AC of triangle ABC is less than the sum of the two other sides, AB, BC.

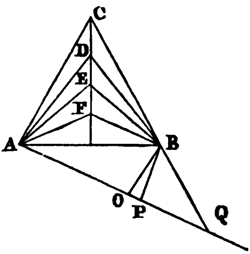
---

[1] The triangle that has its three angles equal is called equiangular.

Fig. 53 represents all the different kinds of rectilineal triangles. The right line AB answers as base to all. Then

ACB is equiangular, equilateral, and acute-angled.
ADB is isosceles and acute-angled.
AEB is isosceles and rectangular.
AFB is isosceles and obtuse-angled.
ABO is scalene and rectangular.
ABP is scalene and acute-angled.
ABQ is scalene and acute-angled.

Fig. 53.



Many public buildings have as façade a front elevation like AFB.

If from each member of the inequality,

$$AB + BC > AC$$

the side BC be subtracted, this gives the new inequality:—

$$AB > AC - BC.$$

The sum of the three angles of every triangle is equal to two right angles. For drawing through one of its summits a parallel to the opposite side (or base), the three angles formed at the same point B, namely m, n, o, are equal to two right angles; but one of these three angles, o is that of the triangle, and the two remaining angles m, n, are equal as alternate angles to the two other angles of the triangle A, C, from which the truth of the proposition is evident.

It is inferred from this that a triangle cannot have two right angles, or two obtuse angles, or one obtuse and the other right.

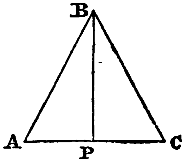
In the right-angled triangle, the side opposite to the right angle is named the hypothenuse, and the two other sides are named the sides containing or adjacent to the right angle.

If two triangles contain two angles of the one, respectively equal to two of the other, the third angles will be also equal.

Equal sides are opposite to equal angles, and reciprocally. (Fig. 55.)

This can be easily determined by dropping the perpendicular BP of the triangle, and doubling the figure across it.

Fig. 55.



Hence the equilateral triangle is equiangular, and the equiangular is equilateral.

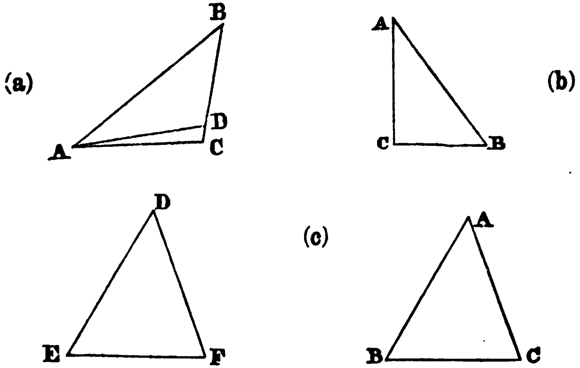
An equilateral triangle cannot be either right-angled or obtuse-angled.

For each of the angles of the equilateral triangle is equal to the third part of two right angles, or the quotient of the division of 180° by 3: a fortiori the equilateral triangle cannot be obtuse-angled; for the other two angles, and therefore the sides subtending them, must be less than the obtuse angle and the side subtending it.

The greater angle is opposite to the greater side; and, recipro-

cally, the greater side is opposite to the greater angle. (Fig. 56, a.)

Fig. 56.



The side AB, being greater than AC, the angle C will be greater than the angle B. Hence the hypothenuse is greater than each of the sides of the right angle. (Fig. 56, b.)

1st. Let angle $CAB >$ angle B; then side CB is greater than side CA. To prove it: make angle $DAB = B$. Then triangle DAB, having angles B and A equal, will be isosceles, and side $AD = DB$. But the straight line $CA < CD + DA$. Substituting DB for DA, as it is equal to it, we have $CA < CD + DB$, or $< CB$ (Fig. 56, a).

2nd. If the side CB be supposed $> CA$, angle $CAB >$ than angle B; for, if not, angle CAB would be either $=$ or $< B$. But if CAB were $= B$, side CB would be $= CA$: and if angle CAB were less than B, side CB would be (by case 1) less than CA. As this is absurd, $CAB > B$. (Fig. 56, a.)

§ XVI.

Equality of triangles. To construct a triangle knowing its following elements, the three sides, two sides and the included angle, one side and the adjacent or contiguous angles. To construct a right-angled triangle, the hypothenuse and one side being given; or the hypothenuse and an acute angle, a side and an acute angle, or the two sides of the right angle being given.

70. The name of equal figures is given to those which, when superposed, coincide throughout; hence all their elements are in all respects equal.

Triangles, owing to their special properties are *equal*, when they have the three sides equal, two sides and the included angle equal, or one side and the adjacent angles equal. (Fig. 56, c.)

1. Let ABC, DEF be two triangles, having: $AB = DE$, $AC = DF$, $BC = EF$. Then angle $A = D$; for if they were unequal, sides BC and EF would be unequal. Therefore $A = D$.

The equalities $AB = DE$, $AC = DF$, $BC = EF$, lead to $A = D$, $B = E$, $C = F$.

2. Let ABC, DEF be two triangles, in which you have: $C = F$, $CA = FD$, $CB = FE$.

Superposing triangle DEF on ABC, so that FD coincides with CA, angle F being equal to angle C, FE will take direction CB; and as $FE = CB$, point E will fall on point B. Therefore the triangles coincide.

The equalities, $C = F$, $CA = FD$, $CB = FE$, lead to $A = D$, $B = E$, $AB = DE$.

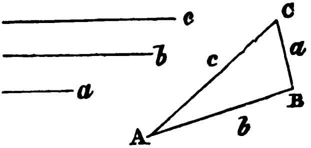
3. In triangles ABC, DEF let $AB = DE$; angle $A =$ angle D; angle $B =$ angle E.

Placing DEF on ABC, so that DE coincides with AB; angle D being $= A$, DF will take direction AC. Similarly, angle E being $= B$, side EF will take direction BC and point F will fall on the line BC. Then, as point F must fall both on AC and on BC, it must coincide with C. Therefore the triangles coincide and are equal.

71. To construct a triangle, knowing the following elements.

1. *Three sides.*—Describing from the extremities of one of them, two arcs with radii equal to the other two sides, their points of intersection will be the third vertex or summit of the triangle. (Fig. 57.)

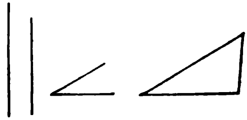
Fig. 57.



2. *Two sides and the included angle.*

Taking on the sides of the given angle, lengths equal to the given sides, we shall have the remaining summits of the triangle required. (Fig. 58.)

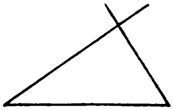
Fig. 58.



3. *One side and the two adjacent angles.*

Constructing at the extremity of a known straight line, two angles respectively equal to those given, the problem will be solved. (Fig. 59.)

Fig. 59.



In the first and third case, it may happen that the sides and angles given are such that the problem cannot be solved.

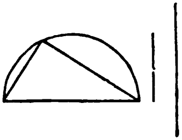
Thus, in Case 1, if the sum of two sides be not greater than the third side.

In Case 3, if the adjacent angles be rights, or if one be right and the other obtuse.

72. To construct a right-angled triangle, whose sides are:

1. *The hypothenuse and one side of the triangle.* Drawing a circumference on the hypothenuse, and taking from one of its extremities a chord equal to the side known, the chord uniting it to the other extremity of the diameter will solve the problem. (Fig. 60.)

Fig. 60.



2. *The hypothenuse and an acute angle.* Describing a circumference on the hypothenuse and drawing from one of its extremities a chord that forms with it an angle equal to the given angle, the other summit of the required triangle will be determined. (Fig. 61.)

Fig. 61.

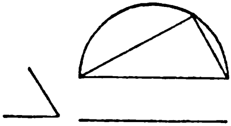
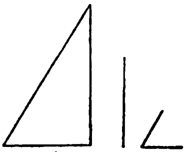
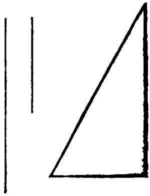


Fig. 62.



3. *Given a side and an angle.* Forming at the extremities of the known side two angles, one right and the other equal to the given angle, the problem will be solved.

Fig. 63.



4. *Given the two sides.* Take on the sides of a right angle, and beginning from the summit, two straight lines equal to the proposed sides, and we shall easily obtain the required triangle. (Fig. 63.)

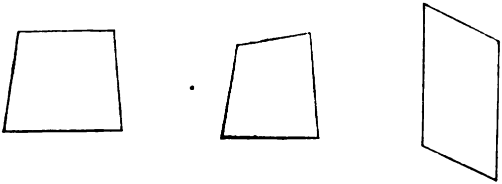
§ XVII.

The quadrilateral may be a trapezoid, a trapezium, and a parallelogram. The square, rhombus, rectangle, and rhomboid. Sum of the four angles of a quadrilateral. Properties of the parallelogram.

73. The quadrilateral is a figure enclosed by four lines called sides.

The diagonal is the straight line which unites one vertex with the opposite one.

Fig. 64.



Trapezium. *Trapezoid.* *Parallelogram.*

The quadrilateral is divided into

(*a*) A trapezoid, when it has no side parallel with another.

(*b*) A trapezium, when it has two sides parallel [2].

(*c*) A parallelogram, when the four sides are parallel, two to two.

One of the diagonals of a quadrilateral divides it into two triangles; and the two diagonals divide it into four triangles.

The parallelogram is divided into (Fig. 65) the following figures:

A square, if it has equal sides and if its angles are right angles.

A rhombus, if it has equal sides and the angles are not right angles.

A rectangle, if the sides are unequal and its angles are right angles.

A rhomboid, if the sides are unequal and the angles are not right angles [3].

The diagonal of a parallelogram divides it into two equal triangles; for they have a side common to both, viz. the diagonal, and the angles adjacent to this side equal as being alternate angles [4].

---

[2] The parallel sides of a trapezium are called the bases of the trapezium, using the term trapezium as in France. See Todhunter's Euclid for the Use of Schools, p. 5 (1867).

[3] Figure 65 represents all kinds of quadrilaterals.

[4] In the square, these triangles are isosceles and right-angled. In the rectangle, they are scalene and right-angled; in the rhombus, they are isosceles, acute-angled, and obtuse-angled; and in the rhomboid, they are scalene, acute-angled, or obtuse-angled.

74. The sum of all the angles of a quadrilateral is equal to four right angles.

For tracing a diagonal, the quadrilateral is divided into two triangles, whose angles form those of the quadrilateral; and as the three angles of every triangle are equal to two right angles or 180°, the angles of the quadrilateral must be equal to four right angles or 360°.

From this it follows that a quadrilateral cannot contain three obtuse angles and one right angle, or two obtuse angles and two right angles, or one obtuse angle and three right angles.

Two quadrilaterals with three angles respectively equal will have the fourth angle also equal.

75. In every parallelogram the opposite angles are equal; and the opposite sides also.

For the opposite angles have their sides parallel, therefore they are equal, and as the opposite sides are parts of parallels intercepted between parallels they are also equal.

Therefore if one of the four angles of a parallelogram is a right angle, the other three angles will be right angles.

The diagonals of every parallelogram cut each other mutually at the centre, those of the square and rectangle are equal, and those of the square and rhombus, or lozenge, cut each other perpendicularly.

This is inferred from the comparison of the four triangles into which a parallelogram is divided at its centre by its diagonals.

§ XVIII.

Equality of parallelograms and of quadrilaterals in general. Construction of squares, rectangles, rhombuses, and rhomboids, knowing some of their elements. To construct a quadrilateral, given three sides and two angles, or three angles and two sides.

76. Squares are equal, if they have an equal side.

Rectangles are equal, if they have respectively two contiguous sides equal.

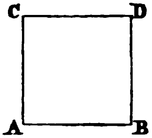
Rhombuses are equal, if they have respectively one side and one angle equal.

Rhomboids are equal, if they have two sides equal and the included angle equal.

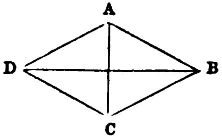
Quadrilaterals are equal, if the triangles into which each can be decomposed are respectively equal in both quadrilaterals, and if they be similarly placed.

77. Construction of a parallelogram in the following cases. (Fig. 65.)

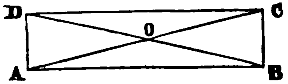
Fig. 65.



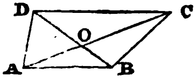
Square.



Rhombus.



Rectangle.



Trapezium.

To construct a square, knowing its sides. At the extremities A and B of the given straight line let two perpendiculars be raised, whereof the length shall be equal to AB; uniting the points C and D, we shall have solved the problem.

To construct a rhombus, the side AB and one angle A being known.

Taking on the two sides of the given angle, the equal distances AB and AD, and describing from B and D with the same radius two arcs cutting each other, the point of intersection C will give us the solution of the problem.

To construct a rectangle, knowing its contiguous sides.

Taking on the sides of a right angle the distances AB and BC equal to the two sides, describing from C an arc with the radius AB, and another from A with the radius BC, the point of intersection will easily give us the rectangle required.

To construct a rhomboid, two sides and the included angle B being known.

The solution of this problem is analogous to that of the previous proposition, with only one point of difference, viz., the substitution of the given acute and obtuse angle for the right angle.

To construct a parallelogram, given the diagonals and the angle formed by them. Let two lines, AC and BD, be described of indefinite length, forming an angle equal to the given angle; and taking from O the distances OA and OC, equal to the half of one diagonal, and the distances OB and OD, equal to the half of the other, we shall have the four summits of the parallelogram required.

Given the diagonals, to construct a square or a rhombus.

The solution of this problem is the same as that of the previous one, keeping in mind the fact that the diagonals of the rhombus and of the square cut each other always perpendicularly, or forming right angles[5].

§ XIX.

Definition of the polygon and names given to different polygons. Equilateral, equiangular, regular, and irregular polygons. Names of the polygons according to the number of their sides. Decomposition of a polygon into triangles. Value of the angles of a polygon.

78. The name of polygon is given to a plane surface terminated by straight lines.

---

[5] To construct a quadrilateral, when three sides and two angles, or three angles and two sides are given.

The simple inspection of Figure 66 is amply sufficient to enable the student to solve this two-fold problem by himself.

Fig. 66.

The sides of the polygon are the straight lines that form it.

The perimeter of the polygon is the sum of all its sides.

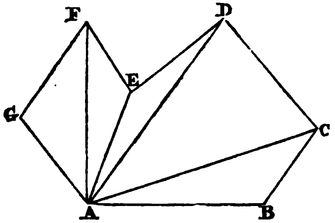
The vertices or summits are the intersections of these sides.

Diagonals are the straight lines that unite two vertices, not immediate or contiguous.

It is evident that in every polygon there are as many angles as there are sides.

The base of a polygon is a side, on which it is supposed to be placed or stand, and its height is the perpendicular, drawn from the most distant vertex from the base to the base or to the prolongation of the base. *AB, BC, CD, DE,* &c., are the *sides* of the polygon (Fig. 67); the broken line

Fig. 67.



ABCDEFGA is its perimeter; the points *A, B, C, D,* &c. are the vertices; the straight lines *AC, AD, AF* are the diagonals; *AB* is the base, and *FA* the height.

The name of equilateral polygon is given to that which has all its sides equal, and the name of equiangular polygon to that which has all its angles equal.

A regular polygon is one that has all its sides equal, and its angles also equal.

An irregular polygon is one that has all its angles unequal.

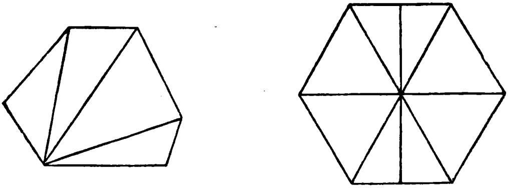
79. Polygons have received different names according to the number of their sides.

The polygon of	three sides	is called a	Triangle.
,,	four sides	,,	Quadrilateral.
,,	five sides	,,	Pentagon.
,,	six sides	,,	Hexagon.
,,	seven sides	,,	Heptagon.
,,	eight sides	,,	Octagon.
,,	ten sides	,,	Decagon.
,,	twelve sides	,,	Dodecagon.

80. Every polygon can be decomposed into as many triangles as it has sides minus two, or into as many triangles as it has sides.

To effect the first decomposition, diagonals are traced from one of the summits to all the others; and to effect the second decomposition it suffices to draw straight lines from any point chosen within the polygon, to all its vertices. (Fig. 68.)

Fig. 68.



81. The sum of the angles of every polygon amounts to as many times two right angles as it has sides *minus* two.

For the angles of the triangles of the first decomposition noticed above composed the angles of the polygon.

If the polygon is equiangular, the value of each angle will be found by dividing the value of all by their number.

Let it be required to find in virtue of this rule, the value of each angle of the regular polygons of, three, four, five, six, eight, ten and twelve sides.

§ XX.

Equality of polygons. To construct on a straight line a regular polygon of any number of sides. To construct a polygon equal to another given one.

82. Equal polygons when superposed ought to coincide throughout their extent, and consequently they have their sides and angles respectively equal, each to each.

From the decomposition of a polygon into triangles, it is inferred that if two polygons can be subdivided into an equal number of triangles, equal each to each and similarly placed, the polygons will be equal.

Let polygons $ABCDE$, $A'B'C'D'E'$ be composed of an equal number of equal triangles similarly placed. Then they are equal; for

1. Their angles are equal, as homologous angles of similar triangles. (See 87.)

2. This similitude gives:—

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}, \quad \frac{AC}{A'C'} = \frac{CD}{C'D'} = \frac{AD}{A'D'},$$

$$\frac{AD}{A'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}, \quad \text{whence comparing we get}$$

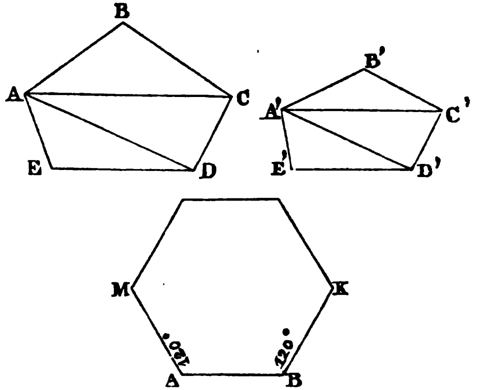
$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}.$$

Hence the polygons are equiangular, and have equal sides; therefore they are similar and equal.

Corollary. If the triangles composing the polygons be equal, each to each, but be not similarly placed, the polygons will not be equal.

83. To construct a regular hexagon on a given straight line.

Fig. 69.



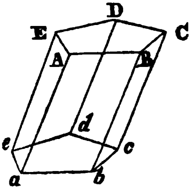
Let AB be the given straight line. (Fig. 69.) At its extremities let two angles be described of 120°, and on their sides lengths equal to AB. At the extremity of these lengths let other angles of 120° be described, whose sides shall again be equal to AB, and continuing in the same manner till we enclose the figure, the problem will be solved.

84. To construct a polygon equal to another given polygon.

The solution of this problem is the same as that of the last, but as the sides and angles of an irregular polygon are irregular or unequal, it is necessary to know all its sides except two consecutive ones, and all the angles except that comprised by the sides neglected.

Another Method. (Fig. 70.)

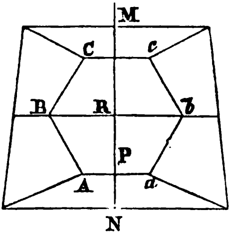
Fig. 70.



Through all the vertices or summits of the given polygon $ABCDE$, let parallels be drawn, and taking equal parts on them from every summit, we shall have the new summits a, b, c, d, e of the new polygon.

Let an indefinite straight line *MN* be drawn; then, dropping from the vertices *A*, *B*, *C*, &c., the perpendiculars *Aa*, *Bb*, &c., so that *AP* may be equal to *Pa*, *BR* to *Rb*, the points *a*, *b*, *c*, will be the vertices of the polygon demanded. The two polygons that result from this construction are called symmetrical in respect of the straight line *MN*.

Fig. 71.



85. The combination of regular polygons is of great use in its application to the arts.

The following examples which the student can solve for himself will serve as an illustration of this remark.

To cover a plane surface with regular polygons.

If the polygons be all equal, the problem can be solved in one of the following ways:—

By squares uniting four angles round each vertex, with equilateral triangles uniting six angles round each summit.

By hexagons, uniting three angles round each vertex.

If the given polygons are of two kinds, the following combinations can be made.

By hexagons and triangles, uniting at each vertex two angles of each of these polygons.

By octagons and squares, uniting at one vertex two angles of an octagon and one of a square.

By dodecagons and equilateral triangles, uniting at each vertex two angles of a dodecagon and one of a triangle.

If the triangles are of three kinds, the problem can be solved by two squares, a triangle, and a hexagon.

§ XXI.

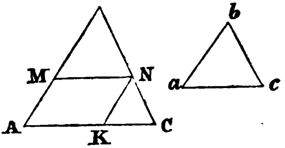
Similar figures. Cases of similarity in triangles. Similarity of parallelograms. Relations of the perimeters and sides of similar figures. To construct triangles similar to other given ones. To construct polygons similar to other given ones.

86. The name of similar figures is given to those that have their angles respectively equal, and their sides proportional.

Another definition of similar figures is this: that they have the same form and a different extension.

87. If by any point of one of the sides of a triangle, a parallel be drawn to the other side, the partial triangle resulting from this is *similar* to the other triangle. (Fig. 72.)

Fig. 72.



Thus, the triangles ABC and MBN are similar, for they have all their angles equal each to each, and their sides proportional.

Every straight line parallel to one of the sides of a triangle divides the two other sides into proportional parts, and reciprocally. Supposing the straight line MN parallel to AC, the student can verify that $BA : BM :: BC : BN$.

Dividing side BA of triangle ABC into three equal parts from B to M, and two equal parts from M to A, this gives

$$\frac{BM}{MA} = \frac{3}{2}.$$

Through the points of division in BA, drawing parallels to the base AC, we also divide BN into three, and NC into two equal parts; therefore $\frac{BN}{NC} = \frac{3}{2}$.

Therefore $\frac{BM + MA}{MA} = \frac{BN + NC}{NC}$; or $\frac{BA}{BM} = \frac{BC}{BN}$.

88. Triangles, in consequence of their special properties, are similar in the following three cases: (1) When they have their three sides proportional to the three sides of the other triangle; (2) two sides proportional, and the angle formed by them equal; or (3) the three angles respectively equal;

that is, the triangles ABC and abc will be similar in each of the following hypotheses :—

1st.—By verifying that $AB : ab :: BC : bc :: AC : ac$.
2nd.—By verifying that $AB : ab :: BC : bc$ and $B = b$.
3rd.—By verifying that $A = a$, $B = b$, $C = c$.

In the latter case, it will suffice that they have two angles respectively equal. (Fig. 73.)

Demonstration of Case 1.

$$\frac{AB}{BM} = \frac{BC}{BN} = \frac{AC}{MN}; \text{ but } \frac{AB}{ab} = \frac{BC}{bc} = \frac{AC}{ac}.$$

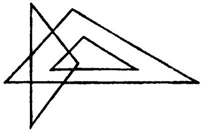
Therefore triangles ABC, abc are similar.

Case 2. $\dfrac{BA}{BM} = \dfrac{BC}{BN}$, and $\dfrac{BA}{ba} = \dfrac{BC}{bc}$. In these two proportions the first three terms are equal; therefore $BN = bc$, and triangle $MBN =$ triangle abc. But MBN similar to ABC; therefore, &c.

Case 3. MBN is constructed similar to abc; NK is made parallel to MA. Then $\dfrac{BA}{BM} = \dfrac{BC}{BN}$. Also $\dfrac{BC}{BN} = \dfrac{AC}{AK}$. Comparing we have: $\dfrac{BA}{BM} = \dfrac{BC}{BN} = \dfrac{AC}{AK}$, or $\dfrac{AC}{MN}$; for $AK = MN$.

Hence $\dfrac{BA}{ba} = \dfrac{BC}{bc} = \dfrac{AC}{ac}$.

Fig. 73.



Two triangles are also similar, if they have their sides parallel or perpendicular each to each; because, on this hypothesis, they will have their angles equal. (Fig. 73.)

89. Parallelograms are similar if they have an equal angle, and its sides proportional; rectangles are similar if they have proportional bases and heights; and rhombuses if they have an angle equal. All squares are similar. (See 75.)

90. Polygons are similar[6] if they can be analyzed into an

[6] Reasoning in 82 establishes this by leaving out the words *equal* triangles, and substituting *similar* triangles with proportional sides.

equal number of triangles, similar each to each, and similarly placed. (See Fig. 69.)

Regular polygons of an equal number of sides are always similar.

91. The perimeters of similar figures are proportional to their sides, and analogous or homologous straight lines containing them. Thus in similar triangles, the perimeters are proportional to their base and height. If one of the sides of a polygon similar to another is 10, 500, or 1000 times larger, &c., than the corresponding side of another, the perimeter of the first will be 10, 500, 1000 and more times greater than that of the second [7].

The relation of the perimeters of two similar polygons $ABCDE$, $A'B'C'D'E'$, is = to that of their homologous sides; for

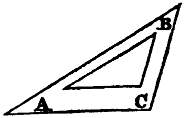
$$\frac{AB}{AB'} = \frac{BC}{D'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}; \text{ whence,}$$

$$\frac{AB + BC + CD + DE + EA}{A'B' + B'C' + C'D' + D'E' + E'A'} = \frac{AB}{A'B'}. \text{ Q.E.D.}$$

92. Given the triangle ABC, to construct another similar to it.

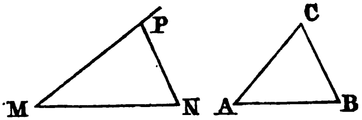
This problem is indeterminate, because you can make as many different triangles as you please, similar to the given triangle; and in order to be similar to it, they need only have their sides parallel or perpendicular to those of ABC. (Fig. 74.)

Fig. 74.



93. On a straight line MN to construct a triangle similar to another given one ABC. (Fig. 75.)

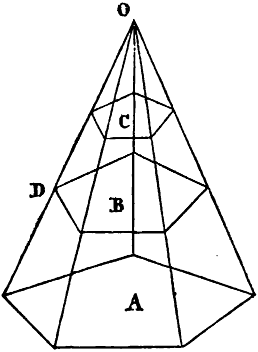
Fig. 75.



Making at the end M an angle equal to A, and at the end N another equal to B, the problem is solved.

[7] Legendre, Éléments, Livre III., Prop. xxx., édition 1862.

Fig. 76.

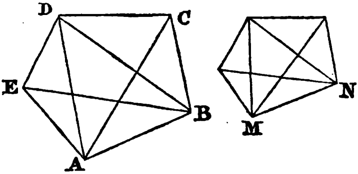


94. To make a polygon equal to the polygon *A*.

Drawing straight lines from all its summits to any point *O*, and then drawing parallels to the sides of the proposed polygon, intercepted by the straight lines, first described, the result will be as many polygons as you like, *B*, *C*, and all similar to the given polygon *A*. (Fig. 76.)

95. On a straight line *MN* to construct a polygon, similar to *ABCDE*. (Fig. 77.)

Fig. 77.



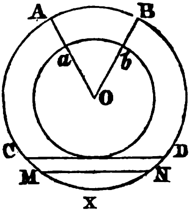
Describing diagonals from *A* and *B*, and forming at *M* and *N* two angles equal each to each, to those of the triangle *ACB*, and two other angles equal to those of *AEB*, the summits of the new polygon will be determined.

§ XXII.

Circular Figures. Polygons inscribed in and described round a circumference. The circumference is the limit of the perimeters of these polygons. Proportion of the circumference to the diameter. Rectification of the circumference.

96. The name of circle is given to the surface comprised by the circumference. (Fig. 78.)

Fig. 78.



A circular sector is the part of the circle terminated by two radii, and the corresponding arc [8].

A circular segment is the portion of the circle comprised between two chords, or between a chord and a corresponding arc [9].

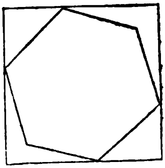
AOB is a circular sector; *CMND* and *MXN* are two circular segments, the first of two bases, and the second of one base.

Circles that have the same radii are equal if the respective radii are equal.

All circles are similar.

97. A polygon is inscribed in a circumference, or a circumference is circumscribed about a polygon, when all the sides of a polygon are chords to the circumference.

Fig. 79.



A polygon is circumscribed about a circumference, or a circumference is inscribed in a polygon, when all the sides of the polygon are tangents to the circumference.

Figure 79 represents a hexagon inscribed in a circumference, and a quadrilateral circumscribed about the same.

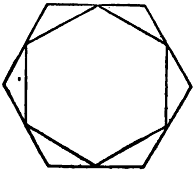
---

[8] "Un secteur est la portion d'un cercle comprise entre deux rayons." Amiot, Éléments, op. cit., 15ième Leçon, p. 56.

[9] "Un segment de cercle est la portion d'un cercle comprise entre un arc et sa corde." Ibid.

98. If a circumference is divided into equal parts, and through the points of division chords or tangents be traced, the inscribed or circumscribed polygon will be regular.

Fig. 80.



In the first case, all its sides are equal, because they are chords of equal arcs, and the angles are also equal, because they are inscribed angles embraced by the same arc. (Fig. 80.)

The same thing is verified in the case when the straight lines are tangents drawn through the points of division of the circumference[1]. And as it is evident that the perimeters of regular polygons inscribed in the same circumference increase, and those of the circumscribed polygons diminish in proportion as you increase the number of sides; it is inferred that the circumference is always greater than each of the inscribed perimeters, and less than any of the circumscribed perimeters, and hence that the circle is the common limit of both. Hence the circumference may be considered as the perimeter of a regular polygon of an infinite number of sides[2].

99. The relation of the circumference to the diameter is the same in all circles[3].

To find this relation, or the number of times that the circumference is greater than the diameter, you inscribe in a circumference of a metre in diameter, a hexagon; after that a polygon with 12 sides, then one with 24, another with 48, another with 96, &c.

---

[1] Amiot, Éléments, op. cit., p. 162.

[2] Manuel du Baccalauréat ès Lettres, par E. Lefranc, and G. Jeannin, Ancien Professeur à l'Académie de Paris, 38ième édition, 1866-67, p. 109. Amiot, Éléments, 28 et 29ième Leçons, Problème vi., p. 147.

[3] Or: circumferences are proportional to their radii. On the mode of valuing the relation of the circumference to the diameter or π, see Note A at the end, which gives the newest methods used in the Polytechnic School at Paris.

When the number of sides is very great, the polygon becomes almost confounded with the circle, and its perimeter with the circumference; therefore by measuring the perimeter of the last polygon, we shall have approximatively the length of the circumference valued at 3·14159 [4], if the diameter be supposed equal to 1.

Accordingly the circumference is a little more than three diameters.

If the name and sign π be given to this decimal quantity, we shall have,

$$\frac{\text{circumference}}{2\,R.} = \pi, \text{ whence circumference} = 2\,\pi R.$$

Archimedes had found for the value of π the fraction $\frac{22}{7}$, or 3·1428. The result can be obtained approximatively by enveloping the circle with a thread, which, when stretched out, will contain the diameter three times *plus* one seventh.

Metius, a Dutch geometrician, at the end of the 16th century, found a much closer estimate for π, making it $= \frac{355}{113} = 3{\cdot}1415929\ldots$

The still exacter methods of modern geometricians have brought out its value to 150 decimals, making $\pi = 3{\cdot}1415926$, which means, that if the diameter is for example one metre, the circumference will have 3·1415 ten-thousandths, only taking the first four decimals.

Given the radius, you will find the corresponding circumference by multiplying it (the radius) by twice the number 3·14159; and given the circumference, you will find the radius, dividing its length by twice the number 3·14159 [5].

100. Rectification of the circumference.

The graphic construction which gives us the length of the circumference in a straight line, is the following. (Fig. 81.)

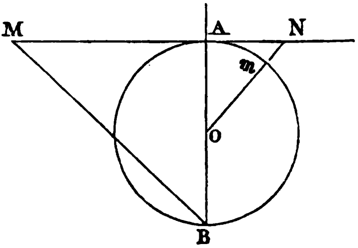
---

[4] More exactly, 3·1415926535. Amiot, p. 148.

[5] The following proportion is also very important:

360 : number of degrees in an arc :: 2 π R : to the longitude or length of the said arc.

Fig. 81.



Let the diameter AB be described; also the tangent MAN, and the straight line ON, in such wise that the arc Am is equal to 30° or half the arc, of which the radius is the chord; beginning at N, let a length equal to three radii be taken on the tangent, and uniting the extremity M with B, we shall have the straight line MB equivalent to half the circumference [6].

§ XXIII.

A triangle can always be inscribed in or described round a circumference. Similarly every regular polygon can be inscribed in a circumference, or described round a circle. Inscribe and describe with a given circumference, the regular polygons of 3, 4, 6, 8, 12, 16, *&c., sides.*

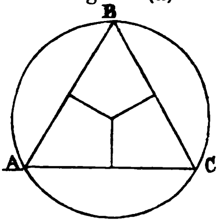
101. Given a triangle, to draw a circumference inscribed in, and another described round it. (Fig. 82.)

---

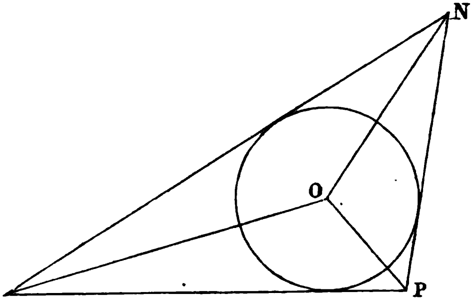
[6] Another estimate is: add to three times the diameter one-fifth of the side of the inscribed square; because the side of the inscribed square $=$ radius $\times \sqrt{2}$. Now one-tenth of the root of 2 is 0·1414, thus adding 3 times the diameter to $\frac{1414}{10000}$ of a diameter, you have almost exactly the quantity required. Géométrie Plane, par L. St. Loup, Professeur à la Faculté des Sciences de Strasbourg, 1868.

The point of intersection of the perpendiculars raised on the centres of its sides will give us the centre of the circumscribed circumference, and the lines bisecting the angles will give us the centre of the inscribed circumference.

Fig. 82 (a).



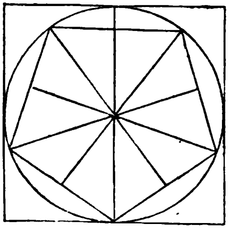
(b)



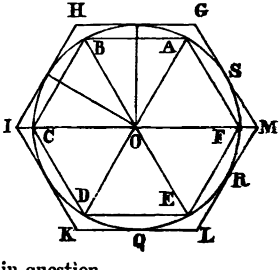
102. Every regular polygon can be inscribed in a circumference, and circumscribed round it. (Figs. 83, a, b.)

Fig. 83.

(a)



(b)



Let *ABCDEF* be the polygon in question.

Through the middle of AB, BC, I raise perpendiculars at K, L, which cut each other at O, the centre of the circumference determined by the three summits A,B,C. To prove that radius OA and line OD are equal, I superpose the two quadrilaterals $OLBA$, $OLCD$, folding the figure over OL. Then the right angles OLB, OLC are equal. The side LC takes the direction LB, point C falls on B, as L is the middle of BC. Similarly, angles LCD, LBA of the regular polygon being equal, as well as its sides CD, BA, CD takes the direction BA, and point D falls on A. Now OD, OA, whose extremities coincide, are equal; therefore the circumference described from centre O, with radius OA, passes through the summit D. It would be proved in like manner that this circumference passes through the other summits of the polygon $ABCD$ . . . this regular polygon can, therefore, be inscribed in a circle.

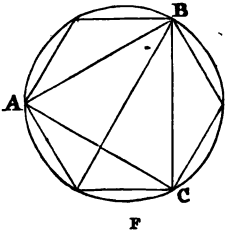
2. The sides AB, BC, &c., of the polygon ABC . . . being equal chords of the circumscribed circle, the perpendiculars OK, OL, &c., let drop from the centre O on these chords, are also equal. Then the circumference described from point O as centre, with the radius OK, passes through the middle, K, L, &c., of the sides AB, BC, and is tangent to each of these lines. Consequently, the regular polygon ABC can be described round a circle.

The name *radius* and *apotheme* of the regular polygon is given to the radius of the circumscribed and inscribed circles. The angles at the centre are equal, because they intercept equal arcs on the circumscribed circumference; therefore the relation of each of these angles to a right angle is $\frac{4}{n}$, n being the number of sides of a regular polygon supposed to be convex.

The first case results from the equality of its radii, and the second from the equality of its *apothemes*, which are described from the centre, perpendicularly to each of its sides.

103. To inscribe an equilateral triangle in a circumference. (Fig. 86.)

Fig. 86.



The radius is taken twice following on the circumference, and the chord of the total arc will be the side of the triangle required.

To inscribe regular polygons of 6, 12, 24, &c. sides, in a given circumference.

To inscribe the hexagon, you take the radius six times in succession on the circumference,

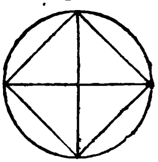
and the chords of these arcs will form the sides of the inscribed hexagon.

To inscribe a polygon of twelve sides, you divide into half the six arcs of the hexagon, and the chords of these new arcs will give us the dodecagon.

In the same manner may be inscribed regular polygons of 24, 48, &c., sides.

104. To inscribe a square in a given circumference. Let two diameters be described perpendicular to each other, and uniting their extremities with chords, we shall have solved the problem. (Fig. 87.)

Fig. 87.



To inscribe in a given circumference, the regular polygons of 8, 16, 32, &c., sides.

To inscribe the octagon, you divide in half the chords of the last problem, and the circumference will be divided into eight equal arcs. The chords of these arcs will be the sides of the required polygon.

In the same manner, polygons of 16, 32, &c., sides can be inscribed.

105. To circumscribe all these polygons round the same circumference; tangents are drawn instead of chords to all the points of division of the circumference, and we shall have the required circumscribed polygons[7].

106. To find the length of the inscribed equilateral triangle in a circumference of which the radius is known.

The radius is multiplied by the square root of three.

---

[7] Proposition vi., Liv. IV., of Blanchet's Legendre. Amiot, Théorème ii., 27ième Leçon. Legendre's proof is by the proportional sides and angles of the inscribed and circumscribed polygons. Amiot demonstrates the same truth by the equal chords and subtending arcs of the two polygons, from which he deduces the regularity of the polygons, which are supposed to be given or already constructed. H. Bos, in his Géométrie Élémentaire, follows Amiot, No. 419, p. 183, 184 (1868).

If the radius of the circle is of 100 metres, the side of the triangle will be equal to

$$100 \times \sqrt{3} = 100 \times 1{\cdot}7320 = 173{\cdot}20.$$

Reciprocally. To calculate the radius of a circle circumscribed round an equilateral triangle.

Divide the side of the triangle by $\sqrt{3}$.

Graphically. Two angles are at the extremity of one of its sides, one a right angle, and the other an angle of 30°; then the hypothenuse will be the diameter of the circle.

107. To find the length of the side of a square inscribed in a circumference.

Let the radius be multiplied by the square root of 2. Formula. $S = R \times \sqrt{2}$.

If the radius of the circle is 100 metres, the side of the square will be equal to

$$100 \times \sqrt{2} = 100 \times 1{\cdot}4142 = 141{\cdot}42 \text{ metres.}$$

Reciprocally. To calculate the radius of a circle circumscribed round a given square.

Let the side of the square be divided by the square root of $\sqrt{2}$.

$$\left(R = \frac{S}{\sqrt{2}}\cdot\right)$$

Graphically. It is sufficient to form at the extremities of one of the sides, two angles of 45°, and the common point of these two straight lines will give us the centre of the circle, and therefore its radius[8].

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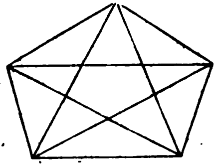
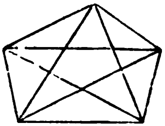
[8] Starlike Polygons. (Fig. 88, a, b.) Given a regular polygon, a

Fig. 88.

(a)



(b)



starlike polygon can be constructed by one of the following pro-

(C.) AREAS OF PLANE FIGURES.

§ XXIV.

The area of a figure. Superficial unity. The area of a triangle, of a parallelogram, of a trapezium, of a polygon in general, regular or irregular, and lastly of a circle. Numerical Problems.

108. The area of a figure is the measure of its superficial extension [9].

The unity of surface, is a square, whose side is a lineal unity, such as a metre, an inch, a foot, &c.

Plane figures that have equal superficial extension are called *equivalent*.

109. *The area of a rectangle* is equal to the product of its base by its height.

This proposition is easily deduced from the simple inspection of Figure 89.

The area of every parallelogram *ABCD* is always equal to the product of its base and height. For it can be directly converted into an equivalent rectangle by bisecting it into equal triangles.

Fig. 89.

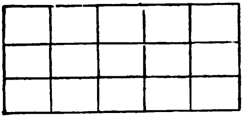
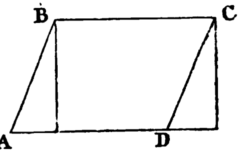


Fig. 90.



Therefore the parallelogram *ABCD* is always equivalent to a rectangle of the same base and height. (Fig. 90.)

---

cesses:—By *reduction* (Fig. 88, a) drawing diagonals not passing through the centre; and by *extension*, prolonging the non-parallel sides of the given polygon (Fig. 88, b). The intersections of these straight lines will give us the vertices of the starlike polygon required.

[9] Amiot, 30ième et 31ième Leçons, p. 156.

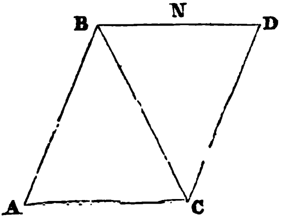
The area of a square is equal to the second power of its side[1].

For being a parallelogram, its area is found by multiplying its base by its height, and as these factors are equal, the product is equivalent to the second power of one of them.

The area of a triangle is equal to the base by half the height.

For the triangle *ABC* is half of the parallelogram *ABDC* of the same base and height as the triangle. (Fig. 91.)

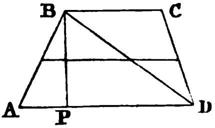
Fig. 91.



The area of a trapezium is equal to the bases multiplied by half the height[2].

For tracing a diagonal, the trapezium will be divided into two triangles whose united areas form it (the trapezium). (Fig. 92.)

Fig. 92.



The area of a regular polygon is equal to its perimeter by the half of its apotheme[3].

---

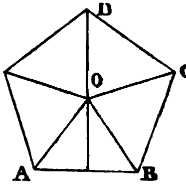
[1] Therefore, to find the length of the side of a square, it will be sufficient to extract the square root of its area. Thus, a field perfectly square, and whose area is 2500 square metres, will have a side of 50 metres; and hence, the length of its boundary or perimeter will be 200 metres, or 2 hectometres.

[2] Also the area of a trapezium is equal to its height multiplied by the parallel half-way between both bases. For this parallel is equal to the base of a parallelogram of the same height and between the same parallels as the trapezium.

[3] Legendre makes the area of a regular polygon equal to its perimeter multiplied by half the radius of the inscribed circle.

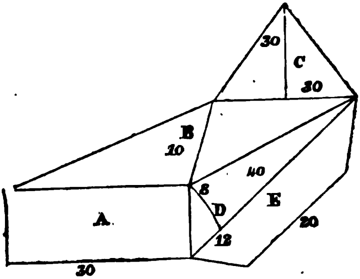
For drawing the radii OA, OB, &c., it will be divided into equal triangles, whose areas make up the polygon. (Fig. 93.)

Fig. 93.



The area of an irregular polygon is equal to the sum of the areas of the triangles, parallelograms, trapeziums, &c., into which the whole of its superficial extension (surface) can be analyzed[4]. (Fig. 94.)

Fig. 94.



The area of a circle is equal to the circumference by half the radius, for the circle can be considered as a polygon of an infinite number of sides.

It is also expressed by the formula πR^2.[5]

(Prop. vii., Liv. IV.) It may be described in other words as equal to its outline multiplied by half the distance from its centre to one of its sides. Géométrie Plane, par L. St. Loup, op. cit., Strasbourg, 1868.

[4] Amiot solves this problem by transforming the given polygon into an equivalent triangle. Leçons 30 et 31, p. 162.

[5] Amiot, p. 183, 34ième Leçon. Legendre, Prop. xi., Liv. IV.

§ XXV.

Numerical problems, relating to the areas of plane figures.

110. The solutions of the following problems relate to the areas of plane figures:—

(1.) How many square feet of carpet are required to cover a rectangular drawing-room, whose dimensions are 20 feet length, and 12 width?

$$20 \text{ ft.} \times 12 = 240 \text{ square feet.}$$

(2.) What is the area of a square place with a side of 120 yards?

$$120 \times 120 = 14{,}400 \text{ sq. yards.}$$

(3.) And if it were a trapezium. Supposing the distance, between both bases, or the height of the trapezium, equal to 200 metres; then,

The greater base $= 150$
„ less base $= 110$

We shall have $260 \times 100 = 26{,}000$ square metres.

(4.) Calculate the extent of a circus, whose radius is 50 metres.

The radius being 50 metres, the circumference will be

$$2\,\pi R = 2 \times 3{\cdot}14159 \times 50 = 314{\cdot}159 \text{ metres.}$$

Consequently we shall have as area of the circle

$$314{\cdot}159 \times 25 \text{ or } 7853{\cdot}97 \text{ square metres.}$$

Otherwise:

$$\pi R^2 = 3{\cdot}14159 \times 50 \times 50 = 7853{\cdot}97 \text{ square metres.}$$

(5.) How many spectators will be held in the said circus, if the radius of the circle of the arena is 20 metres, each person occupying a square metre?

	sq. met.
Total area of the circus	7853·97
Area of the interior ring of the circus	1256·64
Area occupied by spectators	6597·33

Whose number multiplied by 3 will give us the solution of the problem (if there be two tiers as in Spanish bull-rings).

(6.) Find the number of acres of land in a circular piece of land, whose circumference is 2 kilometres and a half.

The circumference being 2500 metres, the radius will be equal to:

$$\frac{\text{Circumference}}{2\pi} = 2500 : 2 \times 3{\cdot}14159 = 397{\cdot}887 \text{ metres.}$$

Hence the area of the piece of ground will be proximatively equal to 497,359 square metres; and reducing this to acres, we have the answer.

(7.) *To find the area of an irregular field,* you divide it into parallelograms, trapeziums, and triangles; and the sum of the areas of these figures will give us that of the entire polygon.

Thus in (Fig. 94) we shall have:

Area of the	rectangle	$A = 600$	square metres.
„	rhomboid	$B = 300$	„
„	triangle	$C = 450$	„
„	triangle	$D = 160$	„
„	trapezium	$E = 360$	„

§ XXVI.

Triangles and parallelograms, of equal base and height, are equivalent. The triangle is half of a parallelogram of the same base and height. The square on the hypothenuse is equal to the sum of the squares on the sides of a right angle. The areas of similar figures are as the squares of their homologous sides.

111. The following points of equivalence are entitled to notice.

Triangles of equal base and height are equivalent.

The triangle is half of a parallelogram of the same base and height.

All parallelograms of equal base and equal height are equivalent.

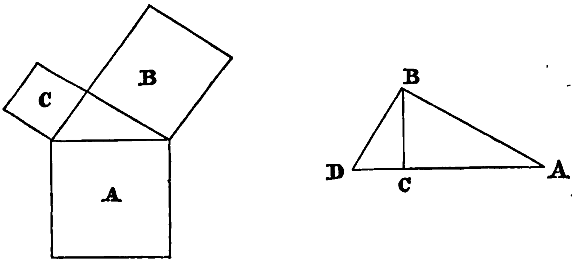
The rhombus is half of the rectangle constructed on its diagonals.

112. The square constructed on the hypothenuse of a right

angled triangle is equal to the squares constructed on the two sides. (Fig. 95, **a** and **b**.)

Fig. 95.

(a) (b)



In Fig. 95 (**b**), by similitude of triangles ABC, BCD, $\frac{AC}{CB} = \frac{CB}{CD}$. (154, Note 8.) Also $\frac{AD}{AB} = \frac{AB}{AC}$, and $\frac{AD}{BD} = \frac{BD}{CD}$. The product of extremes in a proportion equalling that of means, $AD \times CD = BD^2$, and $AD \times AC = AB^2$. Adding this gives:—$AD\,(AC + CD) = AB^2 + BD^2$. $AD^2 = AB^2 + BD^2$.

Hence to make a square twice as large as another square, it is sufficient to construct a right-angled triangle, whose sides are equal to the given square, for in this case, the square constructed on the hypothenuse will be the required square.

To find the numerical value of the hypothenuse of a right-angled triangle, you add the squares of the sides, and from the result you extract the square root.

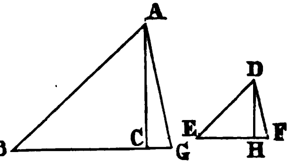
Also the circle constructed on the hypothenuse of a right-angled triangle is equivalent to those constructed on the sides of the right angle.

113. The areas of similar figures are to each other as the squares of the corresponding or homologous sides.

Demonstration.—Case 1. Triangles. Let BAG, EDF be similar triangles. From their summits A and D, drop perpen-

diculars to BG, EF, at the points C, H. Then the triangles BAC, EDH, being equiangular, we have: $\frac{AC}{DH} = \frac{AB}{DE}$ or (Fig. 96)

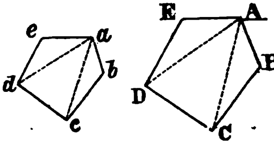
Fig. 96.



halving them, $$\frac{\frac{AC}{2}}{\frac{DH}{2}} = \frac{AB}{DE} \quad (1).$$

Also in triangles BAG, EDF, as they are similar $\frac{BG}{EF} = \frac{AB}{DE}$ (2).

Fig. 97.



Multiplying (1) and (2) we have:
$$\frac{\frac{AC}{2} \times BG}{\frac{DH}{2} \times EF} = \frac{AB^2}{DE^2}.$$

But $\frac{AC}{2} \times BG$ is the measure of triangle BAG, and $\frac{DH}{2} \times EF$ that of the triangle EDF, therefore $\frac{BAG}{EDF} = \frac{AB^2}{DE^2}$.

Case 2. Polygons. Let $ABCDE$, a, b, c, d, e (Fig. 97) be similar polygons. Divide them by diagonals AC, AD, ac, ad into similar triangles.

Then $\frac{ABC}{abc} = \frac{BC^2}{bc^2}$; $\frac{DAC}{dac} = \frac{DC^2}{dc^2}$, $\frac{EAD}{ead} = \frac{ED^2}{ed^2}$.

But as the polygons are similar, we have: $\frac{BC}{bc} = \frac{DC}{dc} = \frac{ED}{d}$,

or squaring $\frac{BC^2}{bc^2} = \frac{DC^2}{dc^2} = \frac{ED^2}{ed^2}$.

Therefore $\frac{ABC}{abc} = \frac{DAC}{dac} = \frac{EAD}{ead}$, and as the sum of the antecedents is to the sum of the consequents, as one antecedent is to its consequent, $\frac{\text{Surface } ABCDE}{abcde} = \frac{BC^2}{bc^2}$.

A square whose side is a yard, or a yard squared, is equivalent to 9 square feet.

A square metre contains 100 square decimetres.

A square whose side is twice another square, and therefore having a perimeter twice as large, will have a surface fourfold as large as the first square. If the radius of a circle is ten times greater than that of another, it will have a circumference ten times larger than the second, but the superficial area of the first will be 100 times greater than that of the other.

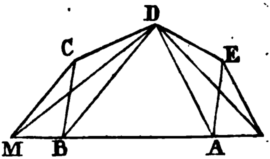
§ XXVII.

To reduce a polygon to another which has one side less. To construct a square equal to a triangle, to a parallelogram, to a trapezium, to a regular or irregular polygon, and finally to a circle. To construct a square or a circle equal to the sum or difference of two others.

114. To reduce a polygon to another equivalent one, having one side less. (Fig. 98.) Prolonging one side, for example, AB; tracing the diagonal BD, the straight line CM parallel to BD and uniting the points D and M, we shall have the polygon $AMDE$, equivalent to the given polygon $ABCDE$; because the triangles BCD and BMD which have the same base BD, and equal heights, are also equivalent.

According to this every polygon can always be reduced to an equivalent triangle. If the polygon be regular, you take as base of the triangle the perimeter of the polygon, as for its height the apotheme[6].

Fig. 98.



115. To transform a parallelogram into an equivalent triangle[7].

The base of the parallelogram may be taken as the base of the triangle, and its height will be double that of the parallelogram. This problem is indeterminate.

116. *Transform a circle into an equivalent parallelogram.*

---

[6] Amiot, op. cit., p. 162.

[7] Géométrie Plane, Saint Loup, No. 272 (1867).

For base you will take the rectified circumference, and for height half the radius. Also an indeterminate problem.

117. The expression, "to square a figure," is used to signify changing it into an equivalent square.

We propose to reduce the following figures to squares:—

1. A triangle. Find a mean proportional between the base and half the height, and we shall have the square required, as both figures will have an equal area.

2. A parallelogram. Let a mean proportional be found, between the base and height, and we shall have the side of the square equivalent to the given parallelogram.

3. A trapezium. Let a mean proportional be found between the sum of both bases, and the half of its height, and we shall also have the side of the required square.

4. A regular polygon. A mean proportional must be found between the perimeter of the polygon and the half of its apotheme[8].

If the polygon be irregular it must be transformed first into a triangle of the same area, and then you find the square equivalent to this triangle.

5. A circle. A mean proportional must be found between the rectified circumference and half the radius.

118. *To construct a square equal to the sum or difference of two other squares.*

6. In the first case, you form a right-angled triangle whose sides are the two sides of the given squares, and the hypothenuse will be the side of the square equivalent to the sum required.

7. In the second case, you form a right-angled triangle, whose hypothenuse will be the larger side of the given figure, and one of the sides its smaller side. Then placing on to the latter, the other side, it will form the side of the square required, equivalent to the difference sought. It is evident that if the given squares are equal, the sum will be equal to the double of each of them.

You solve in the same manner the further problem:—To find a circle equivalent to the sum or difference of two other given circles.

---

[8] Amiot, Problème ii., Leçon 30. Figures Planes.

Cases 1 and 2 (the triangle and the parallelogram) fall into one and are thus solved.

Beginning with 2 (the parallelogram). Let AB be its base, DE its height, and X the side of the square sought. Therefore we should have $X^2 = AB \times DE$, or $\frac{AB}{X} = \frac{X}{DE}$. The side is the mean proportional between AB and DE[9].

Case 1. It would be seen, in like manner, that the side of a square equivalent to a given triangle is a mean proportional between the base and half the height of the triangle.

Case 4 of the trapezium is easily assimilated to that of the parallelogram, and that is met by Nos. 1, 2.

To transform a polygon into a square, the simplest plan is to change it into a triangle and then apply Case 2.

Case 7 of the circles can be compared to that of the squares (118), because circles are conceived to be polygons with an infinite number of sides. It is best determined by the formula $\frac{S}{S'} = \frac{\pi R^2}{\pi R'^2}$, or $\frac{S}{S'} = \frac{R^2}{R'^2}$.

§ XXVIII.

Numerical problems relating to the equivalence of plane figures.

119. Here follow the solutions of certain numerical problems relating to the equivalence of plane figures.

1. What will be the side of a square equivalent to a triangle, whose base is 4500 feet and its height 1000 feet?

The mean proportional between 4500 feet of the base, and 500, the half of the height, is 1500, which shows the number of feet in the side of the square required.

2. It is desired to change a rectangular field for a square one of the same area.

Base of the rectangle 1,000 metres.
Height of ditto 500 „
Mean proportional of these data 707·1.

Hence the side of the square will be 707·1 (to one decimal place) metres.

[9] See also 112.

3. To change the ground plan of an edifice into an equivalent square.

Height of trapezium . .	40	metres.
Larger base of the same . .	150	„
Lesser base of ditto . .	130	„
Sum of both bases . . .	280	„

Therefore the mean proportional between 280 and 20 will be the number of metres in the side of the required square.

4. To change a trapezium whose bases are 100 and height 60 metres into an equivalent circle.

The area of the trapezium *ABCD* (Fig. 65, d.) being equal to that of the two triangles *ABD* and *BDC*, we shall have, area of the trapezium $= 60 \times 50 = 3000$ square metres; and as the circle ought to have this same surface, its radius will be equal to

$$R = \sqrt{\frac{3000}{\pi}} = \sqrt{954{\cdot}9304} = 30{\cdot}9 \text{ metres.}$$

5. What square will be equivalent to a circle whose radius is equal to 1 kilometre, or 1000 metres, and circumference 6283 metres? The side of the square will be equal to 1772 metres.

6. To change the circle of the last given problem into a parallelogram.

This problem is indeterminate, and consequently as many parallelograms can be made as you please; all equivalent to the circle whose radius is 1 kilometre; among others, the following:—

The base of the parallelogram may be 6283 metres, that is to say, the length of the circumference.

In this case, its height will be 500 metres (half the radius).

Another solution:—

Base of the parallelogram 1492 metres (an arbitrary number), and its height will be the quotient, resulting from the division of the area of the circle by 1492.

USE OF ALGEBRA IN THE SOLUTION OF THE PROBLEMS OF GEOMETRY.

Principle of Homogeneity.

Algebra is sometimes employed to solve geometrical problems. In that case, the data and the unknowns are described by letters, and the equations of the problem are written out by using the theorems of Geometry, and applying the rules of Algebra. I suppose that each of these equations is brought to have only rational and whole terms, that is, that the radicals or surds, if any, are made to vanish. But the equation thus transformed is *homogeneous.*

In the demonstration of this important theorem, known as the principle of *homogeneity*, it is regarded as evident, that no relation existing between the lines of a geometrical figure changes with regard to the unity employed to measure these lines.

Thus the square on the hypothenuse of a right-angled triangle is equal to the sum of the squares of the two sides of the right angle, whatever may be the linear unity. It results from this, that no equation of a problem in geometry, expressing a mode of dependence of the unknowns and data of the figure, experiences any modification when you change the size of the linear unity which was used to write down the equation, supposing as previously remarked, that this unity is not one of the given lines.

To establish this principle, I take a case with a difference of lineal unity, and suppose this second unity to be m times less than the first; then the numbers expressing different dimensions of the figure become m times larger, so that each term of the equation ought to be multiplied by m or m^2, or m^3, &c., according as it is of the first, second, or third degree, &c. Accordingly this number will only be independent of the number m, that is, of the variation in the lineal unit, after all its terms will have been multiplied by the same power of m, which requires that they should all be of the same degree.

Construction of the roots of equations of the two first degrees, with one unknown, and of biquadratic equations.

The construction of the roots of an equation only containing one unknown consists in replacing the calculations that would have to be made, to have the value of these roots by a system of graphic operations executed on the given lines, and determining the dimensions of the unknown lines.

To trace these figures, only the ruler and compasses are used. We shall only consider those among these equations, whose roots can be constructed with these instruments, and which do not exceed the first and second degrees, or which are biquadratics.

(1.) Equations of the 1st Degree.

If the equation considered be of the 1st degree, the value of the unknown takes one of the following forms :—

$$x = a \pm b, x = \frac{ab}{c}, x = \frac{abcd}{efg}, x = \frac{abc \pm def}{gh \pm kl}.$$

The constants a, b, c, &c., are here the numerical representations of given right lines.

To construct the first value of x, the length a must be added, or length b subtracted.

The second value of x, is the fourth proportional to the lines c, a, b.

As regards the third value, it is constructed by as many fourth proportionals, as there are linear factors in its denominator. Thus, if you take the two auxiliary unknowns so as to give

$$y = \frac{ab}{e}, z = \frac{cy}{f}, \text{ this gives as result } x = \frac{dz}{g}.$$

Consequently the first thing is to construct the fourth proportional y to the three lines e, a, b, then the fourth proportional z to the three lines f, c, y; and the unknown x will be the fourth proportional to the three lines g, d, z.

Finally, the fourth value of x is brought back to the form of the third, by means of the auxiliary unknowns y and z, chosen so that you have

$$a\,b\,y = d\,e\,f \text{ and } g\,z = k\,l.$$

For this gives: $x = \dfrac{a\,b\,(c \pm y)}{g\,(h\,z).}$

Therefore the student will begin by constructing z and y by means of fourth proportionals. The unknown x is found by the same means, since the factors $c \pm y$, $h \pm z$, are known lines.

(2.) Equations of the 2nd Degree, or Quadratics.

I shall suppose at first that the equation is incomplete, and that it only contains the square of the unknown with a constant term. This equation will take one of the following forms:—

$$x^2 = a\,b,\; x^2 = a^2 \pm b^2.$$

In either case, the unknown has two values, equal and of contrary sign. Therefore it is enough to construct the positive value. It is evident that the positive root of the first equation is a mean proportional between the two lines, a and b, and that the root of the second equation is obtained by constructing the side of a square equal to the sum or to the difference of the squares of the lines a and b.

We will consider next the complete equation of the second degree, which can take one of the following forms:—

$$x^2 - a\,x + b^2 = o \quad (1)$$
$$x^2 + a\,x + b^2 = o \quad (2)$$
$$x^2 - a\,x - b^2 = o \quad (3)$$
$$x^2 + a\,x - b^2 = o \quad (4)$$

Observe first, that the roots of the second equation are equal to those of the first, and of contrary signs; for the second is deduced from the first by substituting in it $-x$ for x. As the fourth equation results from the third by similarly changing x into $-x$, it suffices to explain the construction of the two equations (1) and (3).

We begin by the equation: $x^2 - a\,x + b^2 = o$, and shall call its two roots x', and x'', which we suppose real and therefore positive, since their sum a and their product b^2 are positive. This gives: $x' + x'' = a$, $x'\,x'' = b^2$. These relations convert the question into a known problem, i. e.:—To construct two lines $x'\,x''$, of which the sum and the product

are given. This construction brings to evidence at once the condition of reality in the roots, which is $a > 2b$. As regards the equation: $x^2 - ax - b^2 = o$, of which the roots have contrary signs, we represent the positive root by x' and the absolute value of the negative root by x''. Then: $x' - x'' = a$, $x'x'' = b^2$. Therefore the proposed question becomes the following problem:—Construct two lines, x' and x'', of which the difference and the product are given.

(3.) Biquadratic Equations.

A biquadratic equation can only have one of the four following forms:—

$$x^4 - a^2 x^2 + b^4 = o \ (1)$$
$$x^4 + a^2 x^2 + b^4 = o \ (2)$$
$$x^4 - a^2 x^2 - b^4 = o \ (3)$$
$$x^4 + a^2 x^2 - b^4 = o \ (4)$$

The second of these equations, which is deduced from the first, by substituting $-x^2$ for x^2 has only imaginary roots: therefore we have not to construct these roots. As the fourth equation results from the third by the same change of x^2 into $-x^2$, it is sufficient to show how the real roots of the first and third equations can be constructed. These two equations are, in the first place, reduced to quadratic equations, by making $x^2 = by$, y being here an auxiliary line.

Then the equations in question become

$$y^2 - \frac{a^2}{b} y \pm b^2 = o.$$

Their roots are constructed according to the method shown above, and the unknown x is obtained, by finding a mean proportional between the line b and each of the real and positive values of the auxiliary line y.

Observation. This algebraic method can be applied to the olution of the following problems:

1. Divide a straight line of a givep length in mean and extreme ratio

2. Inscribe a given square in another square also given.

3. Construct a circle tangent at the same time to a circle, to a chord of this circle and to the diameter, perpendicular to this chord. (The radius of the required circle will be reckoned.)

4. Inscribe a rectangle of given area in a triangle.

5. Divide the surface of a triangle into mean and extreme ratio[1] by a perpendicular or a parallel to one of its sides[2].

---

[1] To divide a line of given length into mean and extreme ratio, is to divide it into segments such that the greater is a mean proportional between the less and the whole line.

[2] Amiot, op. cit., pp. 176—181.

PART II.

GEOMETRY IN SPACE AND SOLID GEOMETRY.

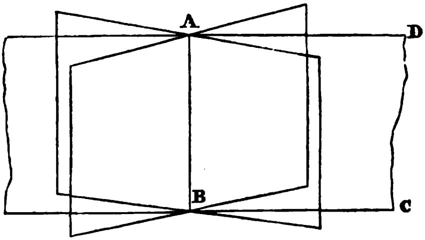
(D.) STRAIGHT LINES AND PLANES.

§ XXIX.

Preliminaries. Conditions of a straight line perpendicular to a plane[1]. *Perpendiculars and obliques to a plane. Distance from a point to a plane. Straight lines parallel to a plane. Projections.*

120. If a straight line has two points in a plane, it will be all in the same plane. Several planes can pass through a straight line[2].

Fig. 99.



The intersections of a straight line and a plane is a point called the foot of the straight line[3].

---

[1] Though, in order to represent planes on paper, we suppose them limited; nevertheless, they ought always to be considered as indefinite or unlimited.

[2] Amiot, Géométrie, Figures dans l'Éspace, lière et 2ième Leçons, Définition 3.

[3] Ibid., lière et 2ième Leçons, Corollaire ii., Théor. i.

Three points, A, B, C, which are not in a straight line, determine the position of a plane. (Fig. 99.)

Because if we conceive a plane to pass through the straight line that unites two of these points A and B, and that it turns round AB till it passes through the other point C, one of the infinite positions of the plane will be determined by the given points. The intersections of two planes is a straight line[4].

The position of a plane is also determined by two straight lines that cut each other, or by two parallel lines. In this manner the plane surface may be considered as engendered (formed or constructed) by the movement of a straight line (generator), pivoting on two other fixed straight lines that cut each other, or are parallel and called directors.

121. A straight line and a plane that cut each other are *perpendicular* to each other, when the straight line is perpendicular to all the other lines drawn in the plane, passing by its foot[5].

A straight line and a plane that cut each other are *oblique* to each other, when the straight line is not perpendicular to all those which drawn through the plane pass through its foot.

A straight line and a plane are parallel, when they never meet, though prolonged indefinitely[6].

122. The straight line perpendicular to two others that cross each other at its foot in a plane, is perpendicular to this plane[7].

For, as has been said, two straight lines that cut each other, determine the position of a plane.

Through a point of a straight line as many perpendiculars as are wished can be drawn to the said straight line, all of which will be found in one and the same plane perpendicular to the given straight line.

---

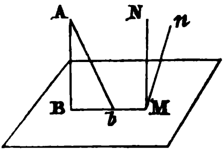
[4] Amiot, Géométrie, &c., Figures dans l'Éspace, Théorème ii., p. 192.

[5] Amiot, Géométrie, Théorème iii.

[6] Amiot, Géométrie, 3ième et 4ième Leçons, Définition 1.

[7] Legendre, op. cit., par Blanchet, 11ième édition, Prop. iii., Liv. V.

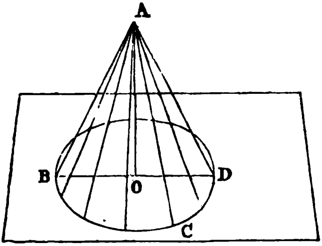
Fig. 100.



123. Through a given point only one perpendicular can be drawn to a plane. For otherwise, you could have a triangle with two right angles or the absurdity of two angles being equal, whereof one is a part of the other. (Fig. 100.)

124. If from a point outside a plane, a perpendicular and different oblique lines are dropped to the same plane, the perpendicular is shorter than all the obliques. (Fig. 101.)

Fig. 101.



The oblique lines deviating the same distance from the perpendicular are equal, and the oblique that deviates the most, is the greatest[8]. (Fig. 103, a.)

Equal obliques can be infinite in number, and at their feet form a circumference, whose centre is the foot of the same perpendicular. (Fig. 101.)

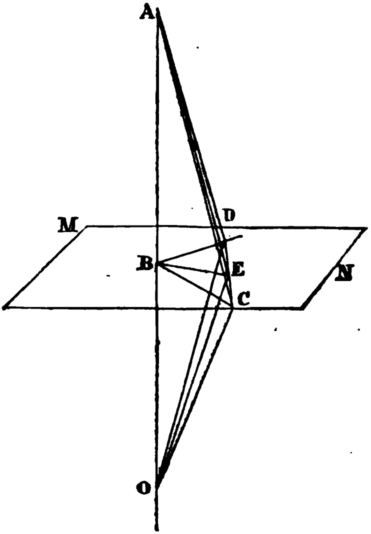
The distance from one point to a plane is the perpendicular drawn from the said point to a plane, because it is the shortest line that can be drawn from the point to the given plane.

Case 1. The perpendicular to two others that cut each other at its foot in a plane is perpendicular to this plane. Let *AB*

[8] Amiot, op. cit., Figures dans l'Éspace, lière et 2ième Leçons, Théorème iii. Proposition vi., Livre V., Legendre, by Blanchet.

(Fig. 102) be perpendicular to BD and BC, passing by its foot in

Fig. 102.



the plane MN. I draw in this plane any third line BE passing by the foot B, and have to prove AB perpendicular to BE. I draw DC and join point A to points C, E, and D. I continue AB to O, making $BO = AB$, and I join point O to points C, E, D. Triangle $CAD = COD$, for as line DB is perpendicular to the middle of A, the obliques DA, DO are equal; in like manner, BC being a perpendicular on the middle of AO, the obliques CA, CO are equal; the third side CD is common; therefore the triangles are equal. Turning triangle COD round CD as a hinge, point O falls on A. Now point E remains invariable; therefore $AE = EO$. Therefore AEO is an isosceles triangle; therefore BE is perpendicular to AO, and reciprocally AO to BE; therefore to plane MN.

Case 2. The oblique that deviates the most from the perpendicular drawn from the same point to a plane is the longest. Let there be two obliques AE, AF, deviating unequally from the

perpendicular AO, and drawn from its summit A. Taking a length $AD = AE$ at the intersection OF of two planes OAF and MN (Fig. 103, a); the oblique line $AE = AD$, because triangles $AOE = AOD$ have side AO common, $OE = OD$ and the included angles equal. But $AF > AD$ (being the enveloping angle); therefore $AF > AE$. Q. E. D. (Fig. 103, a.)

125. Through a given point in space only one line can pass parallel to another given straight line[9].

For parallels must always be in the same plane, in virtue of the very definition of a parallel.

Through a point outside a plane an infinite number of straight lines can pass parallel to this plane: accordingly all these straight lines will be in one and the same plane.

126. The projection of a point on a plane, is the foot of the perpendicular drawn from the same point on to the plane[1]. (Fig. 103, b.)

Fig. 103 (a).

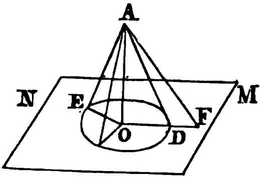
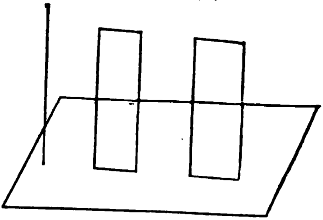


Fig. 103 (b).



The projection of a straight or curved line on a plane, is the line formed by the projections of all its points on the said plane[2].

The projection of a straight line is determined by the projection of its two extremities.

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[9] Legendre, Prop. ix., Livre V.

[1] Legendre, Définition, Prop. xviii., Livre V.

[2] The term "angle" of a straight line with a plane is given to that angle formed by the straight line with its projection on a plane.

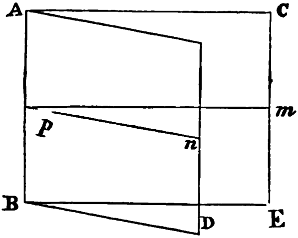
§ XXX.

Dihedral angles. Their magnitude not dependent on the extension of their sides. Adjacent right, acute, and obtuse dihedral angles. Dihedral angles opposed at the common section. Measure of a dihedral angle.

127. A *dihedral* angle is the greater or less inclination of two planes that cut each other. The sides[3] of the angle are the planes that form it, and the common section or ridge[4] is the intersection of these planes[5].

A *dihedral* angle is commonly designated by four letters, of which the first three indicate one of the planes, while the three last designate the other plane, and the letters in the middle designate the common section, ridge, or intersection of both planes. (Fig. 104.)

Fig. 104.



128. The magnitude of a dihedral angle does not depend on the greater or less extension of its sides, but on their greater or less opening or deviation[6].

Two dihedral angles, whereof the sides when superposed coincide, are equal and reciprocally.

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[3] The French name the sides *faces*.

[4] We have adopted the term *ridge* to express the French *arête*.

[5] The *intersection* of *two planes* is always a straight line.

[6] Amiot, 5ième Leçon, Figures dans l'Éspace, Définitions.

129. Adjacent *dihedral* angles are those which have a common section and a common plane, and whose other two sides are in one plane.

If two adjacent dihedral angles are equal, each one is named a *dihedral right angle.*

All *dihedral* right angles are equal. If two adjacent dihedral angles are not equal, the greater is styled *obtuse,* and the less *acute,* and they are *supplementary.*

Dihedral angles opposed at the common section are two angles such, that the sides of one are the prolongations of the sides of the other [7]. These angles are always equal.

130. The measure of a *dihedral* angle is estimated by that of the straight line formed by two perpendiculars to the common section; traced through one and the same point of it, one in one plane, and the other in the other [8]. (Fig. 104, lines *pm, pn.*)

To divide a *dihedral* angle into two, three, &c., equal parts, the corresponding rectilineal angle is divided into the required number of parts, and through the intersection and the sides of these rectilineal angles planes are described [9].

§ XXXI.

Planes perpendicular, oblique, and parallel to each other. If a straight line be perpendicular to a plane, every plane passing through the straight line will be perpendicular to the plane at its foot.

1. If two planes are perpendicular to each other, every right line drawn in one of them perpendicular to their intersection will be contained in the first plane.

2. If two planes are perpendicular every perpendicular raised on one of them at their intersection will be perpendicular to the other.

3. If two planes perpendicular to a third cut each other, their intersection is also perpendicular to the third plane.

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[7] Amiot, 5ième Leçon, Figures dans l'Éspace.

[8] Ibid. Théorème iii., 5ième Leçon.

[9] Manuel du Baccalauréat ès Lettres, Ch. xii., p. 144. Legendre, par Blanchet, Livre VI., Définitions.

4. If a plane meeting another plane makes the adjacent angles unequal, the planes are oblique to each other.

5. Two planes perpendicular to the same straight line are parallel.

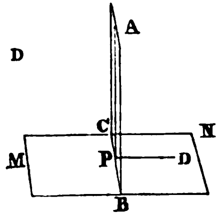
6. If two planes are parallel, every straight line perpendicular to one is also perpendicular to the other.

7. The portions of two parallel straight lines comprised between two parallel planes are equal.

8. Three parallel planes intercept proportional segments on two straight lines which they meet.

Case 1. For if you draw AP perpendicular to the intersection BC of the two planes, it is perpendicular to the plane MN; now from the point A you can only draw one perpendicular to the plane MN (123); therefore it is the line AP contained in the plane ABC. (Fig. 105.)

Fig. 105.



Case 2. Let plane ABC be perpendicular to plane MN, the dihedral angle $ABCN$ being a right angle, its plane angle APD is also right, and the line AP is then perpendicular to the two straight lines BC and PD, which pass by its foot in the plane MN; therefore it is perpendicular to that plane. (Fig. 105.)

Case 3. For if from the point A, common to the two first planes, a perpendicular AB be drawn to the plane DBE, it must be contained in each of the two others; therefore it is their intersection. (Fig. 104.)

Case 4. When a plane, PA, meets another, MN, it forms with it two adjacent angles, $PABM$, $PABN$. If these angles be equal plane, PA is perpendicular to MN; if not, it is oblique to it. (Fig. 106, a.)

Case 5. From the same point one only plane can be passed perpendicular to a straight line. Therefore two planes perpendicular to the same straight line have no common point: therefore they are parallel.

Case 6. Suppose AB perpendicular to the plane M, I say that it is perpendicular to the plane P (parallel to M). Through point A draw in the plane P a straight line AC, and through the straight lines AB and AC I pass a plane, cutting plane M by a straight line BD parallel to AC. Now AB perpendicular to the

Fig. 106 (a).

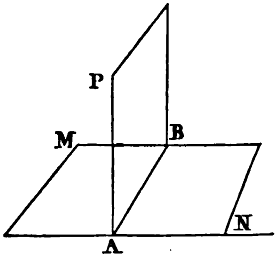


Fig. 106 (b).

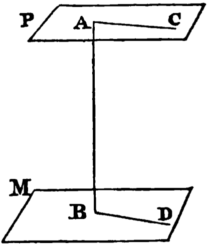
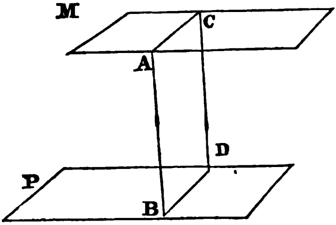


Fig. 106 (c).



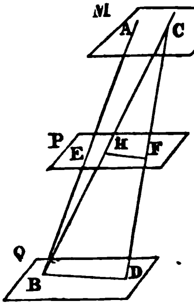
plane M is perpendicular to BD; therefore it is perpendicular to the line AC, parallel to AB. Line AB, being perpendicular to any straight line passing through its foot in plane P, is perpendicular to the plane P. (Fig. 106, b.)

Case 7. The plane of the two parallels AB, CD cuts planes M and P, according to the parallels AC and BD: therefore the figure $ACBD$ is a parallelogram, and $AB = CD$. (Fig. 106, c.)

Case 8. Let A, E, B, C, F, D be the points where the straight lines AB and CD cut the planes M, P, and Q; through the point C draw the line CB which cuts planes P and Q at the points H and B; I join HF and BD; these straight lines are parallel as intersections of the plane CBD by the parallel planes P and Q; this gives $\frac{CH}{HB} = \frac{CF}{FD}$; but $\frac{CH}{HB} = \frac{AE}{EB}$,

as proportionals comprised between parallel planes; therefore $\frac{AE}{EB} = \frac{CF}{FD}$. (Fig. 106, d.)

Fig. 106 (d).



Practically the previous properties of planes are of great use in the industrial arts. Thus most of the walls of our buildings are vertical planes. They are of great use in furniture, doors, windows, shutters, and in perspective[1].

In all these cases it is very essential to determine if a given vertical plane is really vertical.

To do this it is usual to apply to the wall or other object in question, a mason's square, and move it till the plumb-line is contained in the plane of the ruler, and coincides with the plane of the wall.

This line is named the plumb-line or a vertical line, and is a prolongation of the terrestrial radius, corresponding to the said point, where the line is suspended.

Every plane perpendicular to the vertical line is styled a horizontal plane.

Every straight line traced in a horizontal plane is named a horizontal line. Every plane passing through a vertical line is styled a vertical plane.

The intersection of two vertical planes is a vertical line; and the intersection of two planes, one vertical and the other horizontal, is horizontal.

The level of the quiet waters of a lake or sea of small extent represents a horizontal plane.

§ XXXII.

Polyhedral or solid angles of many sides. Every polyhedral angle can be analyzed into as many trihedra as it has

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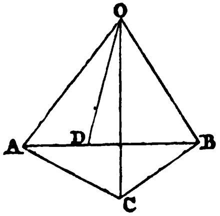
[1] H. Bos, Géométrie Élémentaire, 2ième Année, No. 81.

sides, or into as many as it has sides minus two. Three planes that cut each other divide indefinite space into eight trihedral angles. The plane angles of a polyhedral angle are together less than four right angles[2].

131. A polyhedral angle is the inclination of three or more planes that concur in one point, named the vertex of the angle[3].

The planes composing it are named sides or faces, and their lines of contact are named intersections or *ridges* (*arêtes*) of the polyhedral angle. (Fig. 107.)

Fig. 107.



The planes that unite two intersections of different and opposite sides are named diagonal planes.

A polyhedral angle is composed of as many dihedra as it has sides, and it is named regular when all its dihedral sides and angles are equal

The polyhedral angle that is formed by only three plane angles is styled a trihedron.

132. Every polyhedral angle can be analyzed into as many trihedra as it has sides, or into as many as it has sides minus two.

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[2] Legendre, par Blanchet, Prop. xxxvi., Livre V.

[3] Amiot, Figures dans l'Espace, 7ième Leçon, Définitions.

To effect the first division, it is sufficient to draw planes through any point, chosen inside the polyhedron, to each of its intersections; and for the other division you have to draw diagonal planes from one intersection to all the others.

133. Three planes that cut each other divide indefinite space into six dihedral angles, whereof the common vertex is the point of union of the intersections of the said planes. Fig. 99.)

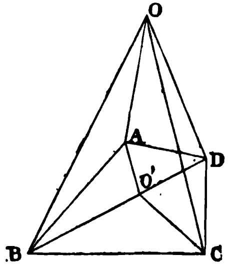
Every plane divides indefinite space into two halves, and each of these is divided by other two planes, into four dihedral angles.

134. In every polyhedral angle, the following properties can be verified :—

1. One side is less than two others; $AOB - BOC + COA$. (Fig. 107.)

2. The sum of its plane angles is less than four right angles, and by consequence, the sum of the angles AOB, BOC, COD and DOA is always less than 360°. (Fig. 108.)

Fig. 108.



For joining all the summits of the polygon forming the base of the polyhedral angle O, to a central point P, I have an equal number of plane triangles at P, and polyhedral triangles at O. (Fig 109.)

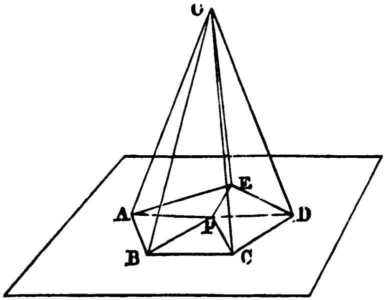
But in the trihedral angle $AOBE$ (Fig. 109), the face BAE is smaller than the sum of the other two [4], thus :—

[4] "Chaque face d'un angle trièdre est moindre que la somme des deux autres." Théorème xcvi., Manuel du Baccalauréat ès Lettres, op. cit.

$$BAE < BAO + EAO$$
$$ABC < ABO + CBO$$
$$BCD < BCO + DCO, \text{\&c.} \quad \text{(Fig. 109.)}$$

and if you add all these inequalities, it leads to the conclusion that the sum of the angles, at the base of the triangles grouped round the point O, is greater than the sum of the angles at the base of the triangles formed round the point P; therefore, by compensation, the sum of the angles formed round O is less than the sum of the angles formed round point P, that is, less than four rights. Q. E. D. (Compare longer proof in Euclid. XI. 22.)

Fig. 109.



(E.) POLYHEDRAL BODIES.

§ XXXIII.

Polyhedral bodies, vertices, faces, ridges or intersections, diagonals and diagonal planes of a polyhedron. Regular and irregular polyhedra. Their names resulting from the number of their sides.

135. The term polyhedral body, or simply polyhedral, is given to the space terminated by plane surfaces.

The sides or faces of a polyhedral are the planes that form it[5].

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[5] "On nomme polyèdres des corps terminés de toutes parts par des plans."—Ibid. p. 144, chap. xii.

The ridges are the sides of its faces, and the vertices are the points of intersection of the different faces.

The *diagonal* is the straight line which unites two vertices of different faces, and diagonal planes are those that pass by three vertices which are not in one and the same face, or those that unite two ridges (intersections) of different faces.

The name *base* of a polyhedron is given to any of its faces, but this term (base) is generally given to that face on which the polyhedron is supposed to rest or stand.

The *lateral faces* are all the faces, except the base or bases, and the lateral ridges are the intersections of each of the lateral faces.

Polyhedra are divided into concave and convex. The second or convex polyhedra are those whose surface can only be cut by a right line in two points.

A *regular polyhedron* is one that has all its faces regular and equal polygons, and also all its solid angles equal.

136. Polyhedra take different names according to the number of their faces. They are called *tetrahedra* if they have four faces; pentahedra, if they have five; octahedra if they have eight; dodecahedra, if they have twelve; and icosahedra if they have twenty faces.

137. The *lateral area* of a polyhedron is the sum of the areas of its lateral faces, and *total area* is the sum of the areas of all its faces.

Two Polyhedra are equal, when they have all their elements equal, and when placed and superposed one on the other, they coincide entirely throughout their extent.

§ XXXIV.

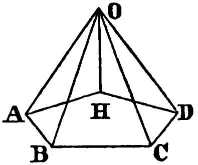
Definition of the Pyramid. Names of its elements. Regular and irregular pyramids. Apothemes of regular pyramids. To analyse a pyramid into tetrahedra. Their lateral and total area. Development into a plane of the lateral and total area of a pyramid.

138. The name *pyramid* is given to a polyhedron terminated by a polygonal base, and as many triangles as the base has sides. (*AOB, BOC, COD, DOA*, Fig. 106.)

The sides of the polygonal base (*AB*, *BC*, &c.) are viewed as bases of triangles, and the common vertex of these triangles is called the vertex or apex (*O*) of the pyramid. (Fig. 106.)

The height of a pyramid is the perpendicular drawn from the apex to the base. (*OH*.)

Fig. 106.



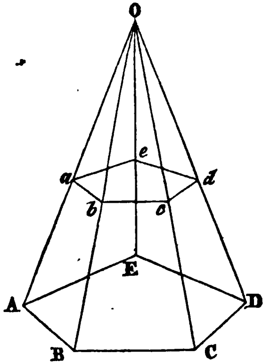
A pyramid takes its name from the nature of the polygon forming its base. Hence it is named triangular, quadrangular, pentagonal, &c., according to the shape of its base.

A pyramid is regular when its base is a regular polygon, and all the lateral ridges are mutually equal [6].

If any of these conditions be wanting, it is called an irregular pyramid.

OABCDE is a regular pyramid. (Fig. 107.)

Fig. 107.



The lateral faces of a regular pyramid have the same height. This height is called the apotheme of the pyramid.

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[6] Legendre, par Blanchet, Livre VI. Définitions xi. xii., xiii.

The term truncated pyramid (or frustum) is applied to a polyhedron which results from cutting a pyramid by a plane parallel to its base, and comprised between the said base and the secant plane.

ABCDE, *a b c d e* is a truncated pyramid.

139. Every pyramid can be analysed into as many tetrahedra as the polygon has sides to its base, or into as many minus two.

In the former case, it suffices to unite the vertex of the pyramid with any point of its base, and to draw through this straight line and each of the summits of the base secant planes [7].

In the second case, the base is divided into as many triangles as it has sides, minus two; and then you draw planes through these straight lines and the vertex of the pyramid.

140. The lateral area of a regular pyramid [8] is equal to the perimeter of its base by the half of its apotheme.

The lateral area of the trunk of a regular pyramid is equal to the perimeter of its bases by the half of the corresponding apotheme.

The total area is found in both cases by adding to the lateral area that of the base, or of the bases respectively.

If the pyramid or truncated pyramid be irregular, its area will be found by summing that of all its faces.

The surface of every regular pyramid or truncated pyramid can always be developed on a *plane.*

141. Find the area of a regular pyramid whose apotheme is of twelve metres; the side of the equilateral triangle answering as its base being two metres and a half.

Since the perimeter of the equilateral triangle that answers as base to the pyramid is equal to $7\frac{1}{2}$ metres, we shall have

Lateral area of pyramid	$7{\cdot}5 \times 6 =$	45 square metres.
Area of its base [9]	$2{\cdot}5 \times 1{\cdot}082 =$	2·71 „
Total area of the pyramid		47·71 square metres.

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[7] Amiot, Figures dans l'Espace, 10 et 11ième Leçons, Théorème v.

[8] That is, *the area* of *its lateral faces.*

[9] The height of this equilateral triangle is found by multiplying half of its side by $\sqrt{3}$; for it is on one side of a right-angled triangle, of which the hypothenuse and the other side are known.

142. Find the area of the truncated part or section of a hexagonal pyramid, whose data are the following:—

Apotheme of the pyramid, 10 metres; side of the greater base, 5 metres; and its apotheme, 433 centimetres.

Side of the lesser base, 2 metres; and its apotheme, 173 centimetres.

Therefore, the perimeter of the greater base will be equal to 30 metres, and that of the lesser base to 12 metres.

According to this we shall have

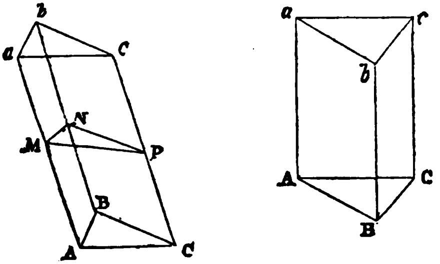
Lateral area of the polyhedron	210	square metres.
Area of greater base	64·95	„
„ of lesser base	10·38	„
Total area of the polyhedron[1]	285·33	square metres.

§ XXXV.

Definition of the Prism. Parallelepiped[2] and cubes. Division of a prism into triangular prisms. Rectangular and oblique prisms. Regular and irregular prism. Lateral and total area of a prism. Projection on a plane of the lateral and total surface of a prism.

143. The name of *Prism* is given to the polyhedron which has for bases two equal and parallel polygons, all the other sides being parallelograms. (Fig. 108.)

(a) Fig. 108. (b)



[1] For the volume of a truncated pyramid, see § xliv. at the end.

[2] The word parallelepiped is thus written in Dr. J. Thomson's Euclid (Second Edition, 1837). Other English mathematicians spell the word parallelopiped, while all French geometricians spell it parallelipiped. I have adopted Dr. Thomson's spelling.

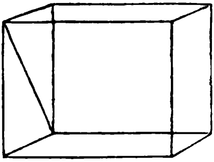
The height of the prism is the distance between its bases. The lateral ridges (intersections) of a prism are equal. The name of rectangular prism is given to the prism that has all its lateral intersections perpendicular to the bases. (*Aa*, *Cc*, Fig. 108, **b**.)

An oblique prism is one whose lateral intersections (*Aa*, *Cc*) are oblique to the planes of both bases. (*ABC*, *abc*, Fig. 108, **a**.)

A prism is called regular when it is rectangular, and its bases are regular polygons; and it is called irregular if it be wanting in any of these points.

The prism receives different names, according to the shape of its bases; thus there are triangular, trapezial, rhomboidal, and other prisms, according as their bases are triangles, trapezia, rhombuses, &c. When the bases are parallelograms, they are called parallelepipeds, and if these parallelograms are rectangles, the parallelepiped is called rectangular. (Fig. 109.)

Fig. 109.



The cube is a parallelepiped whose six sides are squares.

144. Every prism can be analyzed or dissected into as many triangular prisms as the base has sides, or into as many as it has sides, minus two [8].

Thus, if we divide both bases into triangles respectively equal, and equal in number in each base to the number of sides in each base, the planes cutting it passing through the straight lines homologous or corresponding in one or the other base will give us the first-mentioned analysis of the prism, according to the sides of the base.

The second analysis—according to the sides of the prism, minus two—results from the division of both bases, each into as many triangles as it has sides, minus two, after which you draw secant planes through the lines of division.

The lateral sides opposed to each other in every parallelepiped are equal and parallel.

[8] Un prisme quelconque peut-être partagé en autant de prismes triangulaires de même hauteur qu'on peut former de triangles dans le polygone qui lui sert de base. Legendre, op. cit., Liv. VI., Prop. xii., 3.

145. The lateral area of an oblique prism is equal to the product of one of its common sections or intersections by the perimeter of a section perpendicular to the said intersection. (Fig. 108, a.)

The lateral area of a rectangular prism is equal to the product of one of its lateral intersections by the perimeter of its base.

The total area is found by adding to the lateral area that of the bases.

The lateral or total superficies of every rectangular or oblique, regular or irregular prism, can always be projected on a plane.

146. *Problem in illustration.*—To calculate the metres of fancy paper that will be required to cover the walls of a saloon:—

The shape of the saloon in question being that of a rectangular parallelepiped, its lateral area, or, in other words, the height of its walls will be found by multiplying the perimeter of its base by its height.

Supposing the length of the saloon . . .	15 yards.
And its width	10 „
The perimeter of the floor of the saloon will be	50 „
The height of the saloon is	7 „
Hence the required lateral area is . . .	350 yards.

And supposing that the pieces of fancy paper are 6 in. in width there will be required $\frac{350}{6}$, or 58·3 yards of paper.

147. To find the total area of a regular triangular prism, which has a height of 20 feet, and the side of the base = 144 inches or 12 feet.

The perimeter of the base of this prism being equal to 30 feet, the lateral area will be $36 \times 20 = 720$ square feet; and as the area of each base is $12 \times 5·196 = 63·35$ square feet, the total area required will be equal to: $720 \times 62·35 + 62·35 = 844·70$ square feet.

§ XXXVI.

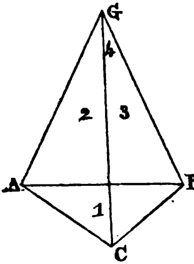
Analysis of a polyhedron into tetrahedra. Regular polyhedra, of which there are only five; the elements common to all of them; their areas. Projection on a plane of the surface of these bodies.

148. Every polyhedron can be analyzed into pyramids, and consequently into tetrahedra. For if we draw secant planes through all its intersections, and any point you please, chosen within the polyhedron, the result will be as many pyramids as there are sides to the figure.

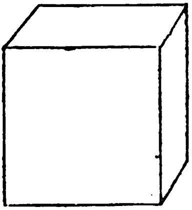
149. We have said that the name of regular polyhedra is given to those that have regular and equal sides, and that have also the dihedral and polyhedral angles equal.

There are only five regular polyhedra; the tetrahedron, the hexahedron, the octahedron, the dodecahedron, and the icosahedron. (Fig. 111.)

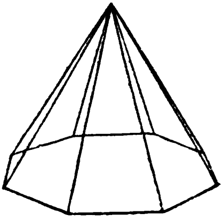
Tetrahedron. Fig. 111.



Hexahedron.



Octahedron.



The tetrahedron is formed of four equilateral triangles. Its angles are trihedra, it has six intersections and four summits.

The hexahedron is formed by six equal squares. Its angles are trihedra; it has twelve ridges or intersections, and eight summits or vertices.

The octahedron is terminated by eight equal equilateral triangles. Its angles are formed by four planes; it has twelve ridges and six summits.

The dodecahedron is formed by twelve regular and equal pentagons. Its angles are trihedra; it has thirty intersections and twenty summits.

The icosahedron is terminated by twenty equal equilateral triangles. Its angles are formed by five planes; it has thirty intersections and twelve summits.

It appears, therefore, that three of these polyhedra are terminated by triangles, one by squares, and another by pentagons.

150. The area of any polyhedron is equal to the sum of the areas of all its sides.

The area of a regular polyhedron is equal to the product of the area of one of its sides by the number of the sides. For as all the sides in such a figure are by its definition equal, it is sufficient to know the area of one of these sides in order to obtain the total area of the polyhedron.

(F.) SPHERICAL OR ROUND BODIES[4].

§ XXXVII.

Definition of a cone. Number of its elements. Equilateral cone. Section of a cone by a plane parallel to its base. Truncated cone. Total and lateral area of a cone. Projection of its lateral and total surface on a plane.

151. The name of *cone* is given to the body formed by the revolution of a rectangular triangle, turning round one of its sides. (Fig. 112.)

The hypothenuse of the generating triangle, or the side of the cone, describes the conical surface, and the other side

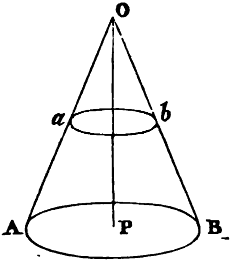
[4] Named by French Geometricians *Les Corps Ronds.*

traces the base. If the side of the cone be equal to the diameter of its base, the cone is called *equilateral.*

Every section made in a cone by a plane parallel to the base, ($a\ b$) is a circle less than the base; and every section that passes through the axis is double of the generating triangle.

A segment of a cone is the part comprised between the base and another plane which cuts all the sides. ($Aa\ Bb$).

Fig. 112.



If the base and the secant plane are parallel, the segment of the cone is one with parallel bases.

$A\ a\ b\ B$ is a segment of a cone with parallel bases.

152. The *lateral area of a cone* is equal to the circumference of the base by half the side.

For the cone may be considered to be a regular pyramid of an infinite number of sides infinitely small.

The formula that expresses the lateral area of a cone is πRS, which means the product of the constant number 3·14159, by the side of the cone, and by the radius of its base [5].

153. *The total area* is found by adding the area of the base to the lateral area.

The lateral area of a segment of a cone with parallel bases is equal to the circumferences of its bases by half its side [5].

Its total area is found by adding the area of both bases to the lateral area [5].

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[5] Finck, Géométrie Élémentaire, Livre VII. p. 241, Prop. xviii. Formula for the convex surface or total area of the cone, r being radius of base and s the side of the cone: $\frac{1}{2}\ s \times 2\pi r$ or πr. s. The surface of the cone may also be obtained by multiplying its side by the circumference of the section, made at equal distances from the summit and the base. For surface $ACMB$ = $AB \times \frac{1}{2}$ circ. DB. Now, if through the point E supposed to

For the truncated cone may be regarded as a truncated pyramid with parallel bases, and the measure of this truncated pyramid is the height of one of the lateral trapezia by the half sum of the perimeter of its two bases.

Let h be the height of the lateral trapezium, b and b' its two bases, and n the number of trapezia, $n\,b$ and $n\,b'$ will be perimeters of the two bases of the truncated pyramid. Then—
Lateral Area or LA of the truncated pyramid or $T\,P$, that is, $L\,A$ of $T\,P = h \times \frac{n\,b + n\,b'}{2}$; area of one of the lateral trapezia $h.\,\frac{b + b'}{2}$; hence as all the trapezia are equal, LA of $TP = h + \frac{b + b'}{2} \times n = h \times \frac{n\,b + n\,b'}{2}$.

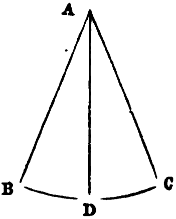
As [6] the Cone is (by hypothesis) equilateral, its lateral and total areas will be expressed by the formula—

$$2\,\pi\,R^2 \text{ and } 3\,\pi\,R^2.$$

The development of the lateral surface of a cone on a plane is a circular sector, whose radius AD is the side of the cone, and its base BC an arc of the same longitude as the circumference of the cone. (Fig. 113.)

The projection of the total surface is obtained by adding the circle of the base of the cone to the previous sector.

Fig. 113.



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be the middle of AB, a section E, G, F, be described parallel to the base CMB, this gives : Circ. DB : Circ. HE :: DB : HE :: AB : AE :: 2 : 1. Therefore Circ. $HE = \frac{1}{2}$ circ. DB, and surface of cone $= AB \times$ circ. HE. (See Note C at the end.)

6 Let L be the middle of the side EB. From this point L, let us draw LQ parallel to BI and the plane LP parallel to the bases of the truncated cone. Trapezium $BIKE$ has also as its measure $BE + LQ$; and it can be proved that LQ is equal to the circumference LO; therefore the surface of the truncated cone has also for measure $BE \times$ circ. LO or $2\,\pi, BE, LO$ (See Figure accompanying Note C at the end).

154. To find the lateral area of a cone which has the following dimensions:—

Side, 15 metres, and the radius of its base 10 metres.

The circumference of the base will be $2\pi R = 2 \times 3{\cdot}14159 \times 10 = 62{\cdot}8318$ metres.

Hence the lateral area of the cone will be $= 471{\cdot}24$ square metres.

Adding to this the area of the base, or 314·16 square metres,

The total area will be 785·40 square metres.

155. To calculate the total area of the segment of a cone with parallel bases whose dimensions are:—

Radius of the lower base	10 metres.
„ „ superior base	5 „
Side of the conic segment	22 „

The circumference of the greater base will be: $20 \times 3{\cdot}14159 = 62{\cdot}8318$ metres.

And the circumference of the lower base $10 \times 3{\cdot}14159 = 31{\cdot}4159$ metres.

Hence the *lateral area* of the conic segment will be found $= (62{\cdot}8318 + 31{\cdot}4159) \times 11 = 1036{\cdot}725$ square metres.

Then as the area of the greater base is $= 3{\cdot}14159 \times 100 = 314{\cdot}159$ square metres,

And the area of the lesser base $= 3{\cdot}14159 \times 25 = 78{\cdot}540$ square metres,

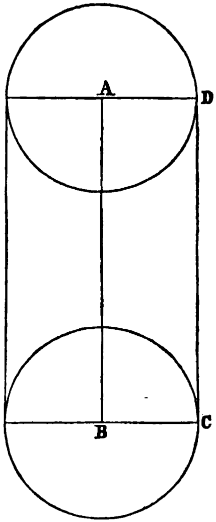
The total area required of the segment of the cone will be the sum of these three last results, or 1429 square metres approximately.

§ XXXVIII.

Definition of the cylinder. Number of its elements. Equilateral cylinder. Section of a cylinder by a plane parallel to the base. Lateral and total area of the cylinder. Development on a plane of the lateral and total surface of a cylinder.

156. A cylinder is a round body, caused by the revolution of a rectangle, which turns round one of its sides, called axis or height. (Fig. 114.)

Fig. 114.



The side opposed to the axis describes the cylindrical surface, and the other two sides trace the coequal and parallel circular bases of the cylinder.

Every section of a cylinder by a plane parallel to the base, is a circle equal to the base; and every section that passes by the axis is double of the generating rectangle.

If the side of the cylinder is equal to the diameter of the base, the cylinder is called equilateral.

157. The lateral area of a cylinder is equal to the product of the circumference of its base by its side or height, because a cylinder may be considered to be a regular prism, of an infinite number of infinitely small sides.

The formula that expresses the lateral area of a cylinder is $2\pi RH$; that is to say, the double product of the constant number 3·14159 by its side and by its radius.

The total is found by adding the area of the bases to the lateral area[7].

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[7] If the cylinder be equilateral, the formulæ of its lateral and total areas are the following:—

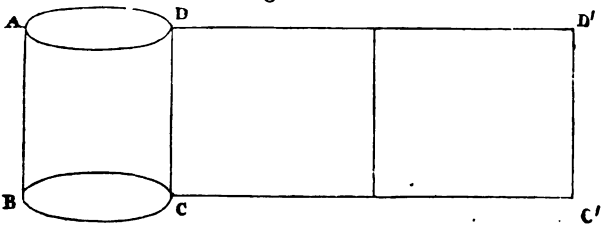
$$4\pi R^2 \text{ and } 6\pi R^2.$$

Lateral area of cylinder *ABCD* can be rolled out into a rectangle *CDD'C'*, whose base *CC'* equals in length circumference of base of cylinder, and whose height *CD* equals that of cylinder.

You make one side of the cylinder *CD* coincide with the plane, and roll it out till *CD* comes to coincide with *C'D'* of the same plane. In this way all the infinite sides of the cylinder will coincide successively with the plane, and the surface of the plane *DC D'C'*, made up of all of them, will be the developed surface of the cylinder. (Fig. 114 B).

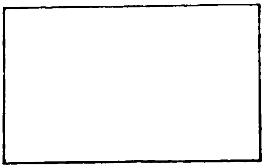
158. The development of the cylindrical surface on a plane is a rectangle whose height is the same as that of the cylinder[8] (Fig. 114 B), and its base the rectified circumference of the base of the cylinder.

Fig. 114 B.



Adding to this rectangle the two bases of the cylinder, we shall have the total development. (Fig. 115.)

Fig. 115.



159. To find the area of a cylinder whose height is 10 feet, and the radius of whose base is equal to 10 inches.

The circumference of the base will be equal to: $2 \times 3{\cdot}14159 \times 10 = 62{\cdot}8318$ inches. The lateral area of the cylinder $=$ 7539·8160 square inches.

§ XXXIX.

Definition of the sphere. Names of its elements. Greater and lesser circles. Section of a sphere by a plane. Plane tangent to a sphere. Area of the sphere. Numerical problems.

160. A sphere is a round body, terminated by a curved surface, whose points are equally distant from another point within it, called the centre. (Fig. 116.)

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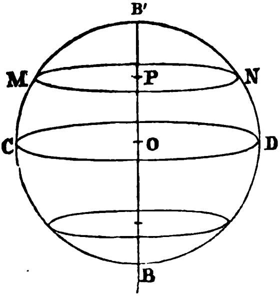
[8] Tratado de Geometria Elemental. By Don Juan Cortasar. Madrid. 12th Edition.

Fig. 116.



A sphere is supposed to be engendered by the revolution of a semicircle, round the diameter, called the axis. (*BOB'*.)

The extremities of the axis are named poles. (*BB'*.)

The radii of the sphere are the straight lines that pass from the centre to the surface. (*OB*, *OB'*, Fig. 116.)

Every section of a sphere made on a plane is a circle.

If the section pass through the centre, it is called a *great circle*, because its radius is that of the sphere. (*COD*, Fig. 116.) If it do not pass through the centre it is called a small circle (*MPN*.) All the great circles of a sphere are equal. Every great circle divides the sphere into two equal parts, which are named hemispheres.

Two spheres are equal when their radii are so.

The name of plane, tangent to a sphere, is given to that which touches its surface in one point alone.

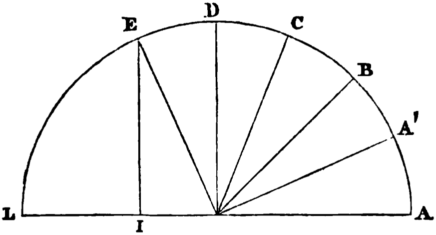
The shortest line that can be drawn on the spherical surface, between two points of the same, is the smallest arc of a great circle, which passes through them.

161. The area of the sphere is equal to the greatest circumference by its diameter, or to four times the area of the great circle.

The area of a sphere is composed of the two zones, engendered by the two axes *AE* and *EL* (Fig. 116 B.), which have as areas $2\,\pi\,r\,AI$, and $2\,\pi\,r\,LI$. Hence the area of the sphere $= 2\,\pi\,r\,AI + 2\,\pi\,r, LI = 2\,\pi\,r\,(AI + LI) = 4\,\pi\,r\,AL$.

Let A be the area of the sphere; then $A = 2\,r \times 2\,\pi\,r$ or $A =$ - r^2.

Fig. 116, B.



This gives two corollaries :—

1. Area of sphere = 4 times that of its great circle, for area of great circle $= \pi r^2$.

2. Area of sphere is equal to the lateral area of the circumscribed cylinder, and is two-thirds of the said cylinder. For the lateral area of the circumscribed cylinder $= 2r \times 2\pi r = 4\pi r^2$, and the total area will be $4\pi r^2 + \pi r^2 + \pi r^2 = 6\pi r^2$; and it is clear that the area $4\pi r^2$ of the sphere is two-thirds of $6\pi r^2$.

The formula for the area of the sphere is $4\pi R^2$; that is, four times the product of the constant number 3·14159 by the square of the radius.

162. The areas of spheres are as the squares of their radii, and consequently if the radius of a sphere is double the radius of another sphere, the area of the first will be four times greater than that of the second; and if the radius of the one were ten times greater than that of the other, the area of the first would be 100 times greater than the area of the latter.

163. Solve the following numerical problems relating to the area of the sphere.

(1.) Find the number of square metres required to make a spherical cloth globe, supposing the radius to be = 10 metres: $4\pi R^2 = 4 \times 3{\cdot}14159 \times 100 = 1256{\cdot}636$ square metres.

(2.) To find the area of the earth in square leagues.

Supposing the earth perfectly spherical, and its radius of 1145 leagues, we shall have as its area

$$4 \times 3{\cdot}14159 \times 1145 \times 1145 = 16474812.$$

(3.) Find the area of the torrid zone of our globe, sup-

posing its height to be 912 leagues. The greatest circumference of the earth amounts to 7200 leagues, therefore the area of the torrid zone, will be equal to $7200 \times 912 = 6,566,400$ square leagues.

In the same manner we shall have, for the area of each of the glacial zones, the product of 7200 by 95, or 684,000 square leagues.

(4.) How many times is the area of the sun greater than that of the earth, knowing that the solar radius is 112 times greater than that of the earth?

The required number will be the square of 112 or 12544.

(5.) Knowing the area of a sphere, to find its radius.

Let the given area $= 100$ square metres. Dividing this number by 4, it will result that the area of a great circle of the sphere is equal to 25 square metres.

Therefore, if we divide this result by π, and extract from the quotient its square root, the problem will be solved.

VOLUMES OF POLYHEDRA AND ROUND BODIES.

§ XL.

Volume of a body. Unity of volume. Volume of parallelepipeds, prisms, pyramids, and of polyhedra in general. Volume of regular polyhedra. Equivalence of the volumes of some polyhedra. Comparison of the volumes of similar polyhedra.

164. The volume of a body is the measure of its extent in three dimensions[9].

The unity of volume is the cube whose side or intersection is the lineal unity, such as a metre, a foot, an inch, &c.

Geometrical bodies that have an equal volume are called *equivalent.*

165. The volume of a rectangular parallelepiped is equal to the area of its base by its height.

This proposition is easily deduced from the simple inspection of Fig. 117.

[9] On appelle volume l'espace occupé par un corps solide. Ch. xiii. p. 149, Manuel du Baccalauréat ès Lettres.

The volume of any parallelepiped is equal to the area of its base by its height.

Because the given parallelepiped can always be changed into another rectangular one equivalent to it, and of equal height, whose base will have the same area as that of the proposed parallelepiped. (Fig. 118.)

The volume of a *triangular* prism is equal to the area of its base by its height. For every parallelepiped can be divided into two triangular prisms equivalent to it and of equal height, whose bases will also be half that of the parallelepiped. (Fig. 119.)

The volume of any prism can be analyzed into other triangular prisms of the same height, and whose bases form together that of the entire given prism.

Fig. 117.

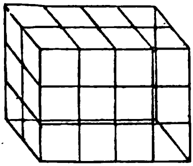


Fig. 118.

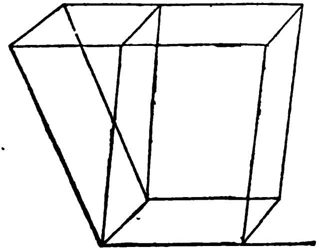


Fig. 119.

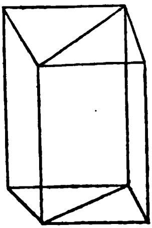
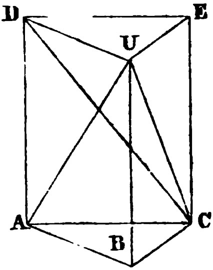


Fig. 120.



The volume of a triangular prism is equal to the area of its base by the third of its height. For every triangular prism can be always divided into three equivalent pyramids. (Fig. 120.)

The volume of any pyramid is equal to the area of its base by the third of its height.

For dividing the pyramid into other triangular pyramids of the same height, the sum of the volumes of these will give us the volume of the entire pyramid.

The volume of any polyhedron is found by dividing it into other polyhedra, whose volume can be determined by the previous rules, and then by taking the sum of the volumes of these partial polyhedra.

The volume of a regular polyhedron is equal to the third of the product of its area by its apotheme.

For regular polyhedra can be considered as formed of regular pyramids whose bases are the faces, and their vertices the centre of the polyhedron.

166. The following points of equivalent deserve to be noted among many others.

Prisms of the same height and equivalent bases are equivalent.

Pyramids of the same height and equivalent bases are equivalent[1].

Every pyramid is the third part of a prism of the same height and equal base.

According to this, it will be easy to reduce an oblique parallelepiped into another rectangular one, and any prism into another of a different base.

Comparison of the volumes of polyhedra :—

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[1] Legendre, Livre VI. Prop. xiv.

The term *similar polyhedra* is given to those that have all their sides respectively similar, and their dihedral and polyhedral angles equal.

Similar polyhedra have their homologous ridges (or intersections) proportional.

Regular polyhedra of an equal number of sides are always similar.

Proofs: (1.) Let R r be homologous ridges of two similar polyhedra, and S, S', S'', the different sides or faces of the one, s, s', s'', those of the other.

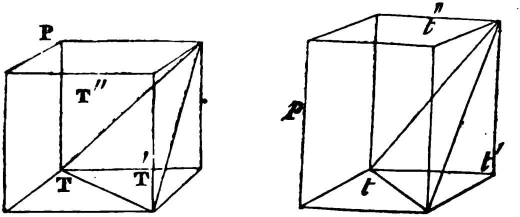
Then: $\frac{S}{s} = \frac{R}{r}$; $\frac{S'}{s'} = \frac{R}{r'}$; $\frac{S''}{s''} = \frac{R}{r}$.

Therefore $\frac{S + S' + S''}{s + s' + s''} = \frac{R}{r}$. (Fig. 121.)

(2.) Two regular equal-sided polyhedra can be analyzed into tetrahedra or pyramids, respectively similar and similarly placed. Let P p be equal-sided polyhedra. Divide them into similar tetrahedra, T, T', T'', and t, t', t'',

Then: $\frac{P}{p} = \frac{N(T + T' + T'')}{N(t + t' + t'')}$. (Fig. 120, b.)

Fig. 120 (b).



(N signifying the number of tetrahedra into which the polyhedra can be divided.)

The volumes of similar polyhedra are to one another as the cubes of their homologous intersections and straight lines; that is to say, that if the intersections of the one are double those of the other, the volume of the first will be 2 × 2 × 2 times greater than that of the second.

If we consider, for example, two similar pyramids having as their bases B b, and their height H h, we shall have $B : b :: H^2 : h^2$, and therefore evidently $\frac{1}{3} H : \frac{1}{3} h :: H : h$.

Hence: $\frac{1}{3} HB : \frac{1}{3} hb :: R^3 : r^3$ (R r signifying homologous ridges).

The volumes of regular polyhedra of the same number of faces, have the same proportion as the cubes of their intersections.

If we consider any two polyhedra, knowing that they are composed of similar tetrahedra, similarly disposed and respectively similar, we shall call them, T, T', T'', and t, t', t'', &c.

As the polyhedra are similar, their homologous intersections or ridges are proportional, or what is the same thing, the proportion of the ridge of one polyhedron to its homologue is equal to the proportion of any other ridge of the first polyhedron to its homologue. Let 9 be this proportion.

$$\text{Then: } \frac{T}{t} = 9^3 \frac{T}{t'} = 9^3 \frac{T'}{t''} = 9^3, \text{ \&c.}$$

$$\text{Therefore } \frac{T + T' + T''}{t + t' + t''} = \frac{T}{t} = 9^3.$$

If the proportion in question were $\frac{5}{7}$, that of the volumes of the polyhedra would be $\left(\frac{5}{7}\right)^3 = \frac{125}{343}$.

If the intersection of a cube or of a regular icosahedron is four times larger than that of another, the volume of the former will be 64 times greater than the volume of the latter.

According to this a cubic metre contains 1000 cubic decimetres [2].

§ XLI.

Numerical problems relating to the volumes of polyhedra.

167. (1.) Find the number of cubic feet of water contained in a reservoir of the following dimensions:—

Its form is a right-angled parallelepiped; and its dimen-

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[2] Demonstrations in Number 166 are identical with those of the Tratado de Geometria Elemental, by Don Juan Cortasar, Examiner for the Madrid University, &c. 12th Edition.

sions are, length 1006 feet, width 440 feet, and depth 12 feet. This will give—

Area of the base, $1006 \times 440 = 442640$ square feet.

And as the height or depth $= 12$ feet, we shall have as the volume of the pond 5311680 cubic feet.

(2.) What is the volume of a triangular pyramid whose base is a hectare, and its height equal to the highest of the three celebrated pyramids of Egypt?

Area of the base $= 10{,}000$ square metres. Height of the pyramid, 146 metres.

Multiplying 10,000 by the third of 146 we shall have the number sought for, which will be proximately $= 486{,}666$ cubic feet.

(3.) Find the volume or mass of a mound of 500 metres in length, supposing that the width at the base is 5 metres, the height $2\frac{1}{2}$ metres, and the width of the upper part 8 metres.

The ground to be measured forms a trapezial prism, whose height is 500 metres, and as the area of its base is equal to the sum of 5 metres and 8 metres, multiplied by the half of $2\frac{1}{2}$, we shall have—

Area of the base, $13 \times 1{\cdot}25 = 16{\cdot}25$ square metres.

Length of the mound or height of the prism, 500 metres.

Therefore the volume required will be 8,125 cubic metres.

If, for every cubic metre, you pay 10*s*. 6*d*., the total cost of this work will amount to 85,312*s*. and 6*d*.

§ XLII.

Volume of the Cone. Volume of the Cylinder. Volume of the Sphere. Equivalence of these volumes. Similar spherical bodies, &c. Comparison of their volumes.

168. The volume of a cone is equal to the product of its base by the third of its height.

For the cone may be regarded as a regular pyramid of an infinite number of faces.

Formula for the volume of a cone:—

$$\frac{1}{3}\ \pi\ R^2 H.$$

Let S s be the areas of two regular infinitesimal and similar pyramids, one circumscribed and the other inscribed in the basis of the cone. The pyramids having as base these polygons, and as summit the summit of the cone, will have as their volume $\frac{1}{3}\ s\ h$ and $\frac{1}{3}\ s\ h$, the common height being h. Now the difference of these two expressions $\frac{1}{3}\ h \times (S-s)$ is infinitely small; therefore the difference between each of these pyramids and the cone is also infinitely small.

Therefore, taking the relation[3] :—

Volume of circumscribed pyramid $= \frac{1}{3}\ S\ h$, and neglecting the infinitely small differences, we have

Volume of the cone $= \frac{1}{3}\ h \times$ area of base. If r is the radius of base, the volume of the cone is $\frac{1}{3}\ \pi\ r^2\ h$.

Inscribe a regular pyramid in a cone. Let p be the area of the base of the inscribed pyramid, h the common height of the cone and of the pyramid, and x the volume of the pyramid,

$$x = \frac{1}{3}\ h p.$$

By sufficiently increasing the sides of the pyramid, B will be the limit of p, and V (or volume of the cone) the limit of x. Then

$$x = \frac{1}{3}\ h\ B.$$

$$V = \frac{1}{3}\ h\ B,\ B = \pi\ r^2,\ V = \frac{1}{3}\ \pi\ h\ r^2.$$

By this equation any of the three quantities, V, h, and r, can be found if the two others are given.

The volume of *a cylinder* is equal to the product of its base by its height. The cylinder is also considered as a

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[3] Géométrie Élémentaire, par H. Bos, Professeur au Lyceé St. Louis à Paris. Deuxième Année, p. 99, No. 172. (Edition 1869). Géométrie Élémentaire, par P. J. E. Finck, Professeur de Mathématiques spéciales dans les Colléges Royaux, Professeur de Mathématiques aux Écoles Royales d'Artillerie. (1841.) Livre VII., p. 243, Prop. xx.

rectangular prism of an infinite number of faces, and consequently the volumes of both bodies are determined in the same manner.

Formula for the volume of a cylinder: $\pi R^2 H$.

Therefore every cone is the third part of a cylinder of equal base and height.

Let V be the volume of a cylinder, B its base, h its height.

Inscribe in the cylinder a regular prism. Let p be the area of the base of this prism, h the common height of the prism and of the cylinder, and x the volume of the prism.

We shall have: $x = ph$. Extending this to the limits of the prism, that is, to the cylinder, we obtain $V = B\,h$.

If r be the radius of the base of the cylinder, B will be $= \pi\ r^2$. Then $V = \pi\ r^2\ h$, a formula from which you can find either V, r, or h, if the two other relations are given.

The *Volume* of a *Sphere* is equal to the product of its area by the third part of its radius[4].

For the sphere is composed of two spherical sectors, $OABD$, $OBCD$ (Figs. 123 and 116), whose zones make up the sphere.

If Z, Z' are the areas of these zones, the volumes of the sectors will be:

$$\frac{1}{3}\ r\,(Z + Z')\ Q.E.D.$$

If V be the volume of the sphere,

$$V = \frac{1}{3}\ r \times 4\,\pi\,r^2, \text{ or } V = \frac{1}{3\pi}\ r^3$$

from which V can be known if r be given.

The sphere may be considered as a regular polyhedron, terminated by an infinite number of faces[5], or composed of

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[4] The volume of the sphere may be obtained in two ways:—1st, By considering it as the sum of an infinite number of pyramids, having for their bases infinitely small plane polygons, confounded with the surface of the sphere, and for their height the radius; or, 2ndly, as engendered by a number of triangles each turning round an axis described in its plane and through its centre.—*Manuel du Baccalauréat ès Lettres*, p. 172, ch. xvii.

[5] Legendre, Livre VIII., Prop. xi.

cones, whose vertices are in the centre of the sphere, and whose infinitely small bases constitute the surface of the sphere.

Formula for the volume of the sphere $\frac{4}{3} \pi R^3$.

169. Comparison of the volumes of round or spherical bodies[6].

The volumes of similar[7] cones and cylinders are respectively as the cubes of their radii, or of their height[8].

(1.) The proportion of the volumes of similar cones is that of the cubes of the radii of their bases or of their height or side. Let R and r be the radii of the bases of similar cones, S and s the sides, H and h the height.

Then by definition of similar cones—

$$H : h :: R : r, \text{ and}$$

$$\frac{1}{3} \pi R^2 : \frac{1}{3} \pi r^2 :: R^2 : r^2.$$

Therefore $\frac{1}{3} \pi R^2 H : \frac{1}{3} \pi r^2 h :: R^3 : r^3 :: H^3 : h^3 :: S^3 : s^3$.

Now as $\frac{1}{3} \pi R^2 H$, and $\frac{1}{3} \pi r^2 h$ are the volumes of the two cones, the theorem has been demonstrated.

Then as regards cylinders—

Let R r be the radii of their bases, H and h their height.

By the definition of similar cylinders—

$$H : h :: R : r$$

$$\pi R^2 : \pi r^2 :: R^2 : r^2.$$

$$\pi R^2 H : \pi r^2 h :: R^3 : r^3 :: H^3 : h^3.$$

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[6] French geometricians style them Les Corps Ronds.

[7] The term *similar* Cones or Cylinders is given to those that have their height and the radii of their bases proportional. If the height of one be half that of the other, the radius of the base of the first ought to be also half the radius of the base of the second. Spheres are always similar.

[8] Legendre, Livre VIII. Corollary 3 makes similar cylinders and cones to each other, as the cubes of the diameters of their bases or of their heights.

The volumes of spheres are respectively as the cubes or third powers of their radii.

If the radius of a sphere is three times greater than that of another, the volume of the former will be twenty-seven times greater than that of the second [9].

§ XLIII.

Numerical problems relating to the volumes of round bodies.

170. The following problems relate to the volumes of round or spherical bodies.

1. Find the volume of a cone of salt whose height is fifteen metres, the radius of its base being twelve metres.

Area of base, $\pi R^2 = 452{\cdot}389$ square metres.

Hence the volume required will be $= 2261{\cdot}945$ cubic metres.

A cubic metre containing 1000 cubic decimetres, and a cubic decimetre being the capacity of a litre, the proposed cone will contain 2261945 litres of salt.

2. Calculate the capacity of a cylindrical well whose diametre is five metres, and its depth a hectometre.

Area of the base, $\pi R^2 = 3{\cdot}14159 \times 2{\cdot}5 \times 2.5 = 19.63494$ square metres, whose number multiplied by the depth of the well will give us the required capacity.

3. Find the volume of the earth in cubic leagues. The formula for the volume of the sphere being $\frac{4}{3}\ \pi R^3$, we shall have $\frac{4}{3}$ of $3{\cdot}14159 \times 1145 \times 1159 \times 1159$ cubic leagues.

4. How many times is the sun larger than the earth?

The proportion of the volumes of two spheres is the same as that of the cube of their radii.

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[9] Let R r be the radii of two spheres $\frac{4}{3}\ \pi R^3$, $\frac{4}{3}\ \pi r^3$ their volumes. It is evident that the proportion of these quantities is the same as $\frac{R^3}{r^3}$.

Corollaire ii., Amiot, Théorème iv. 23ième et 24ième Leçons. Figures dans l'Espace.

Consequently the radius of the sun being 112 times greater than that of the earth, the volume of the sun will be 1,404,928 times greater than that of the globe which we inhabit.

5. What will be the weight in kilogrammes of two spheres, one of gold, the other of silver, supposing that the radius of the former is equal to one decimetre, and that of the second to two decimetres.

Volume of the gold sphere :—

$$\frac{4}{3}\pi R^3 = 4{\cdot}18878 \times 1 \times 1 = 4{\cdot}18878 \text{ cubic decimetres.}$$

And as each decimetre of molten gold weighs 19·258 kilogrammes, the gold sphere will weigh 80·667 kilogrammes.

In the same manner you can reckon the weight of the silver sphere, knowing that a cubic decimetre of this metal weighs 10·474 kilogrammes.

§ XLIV.

The volume of truncated bodies. The lateral and total volume of truncated prisms. Of truncated pyramids with parallel bases. Of truncated c.,nes.

The lateral and total areas of truncated bodies have been considered. (Nos. 142, 154, 155.)

171. We have now to consider the volume of truncated bodies.

In the case of prisms and cylinders no demonstration is required, as the truncated body will evidently bear proportion to the diminution in the height.

The exact value of any truncated prism, and hence cylinder, will be found by dividing it into truncated triangular prisms, taking the volume of each, and then summing them.

172. Truncated pyramids with parallel bases.

The volume of these is found by dividing the truncated pyramid into three tetrahedra, Bh, bh and $\sqrt{Bbh}$, of which Bh will have the lower face as base, $b\,h$ the upper, and $\sqrt{Bb}$ a mean proportional between them. (Fig. 121 and 122.) In Fig. 122, tetrahedron $CDEF$, evidently $= \frac{1}{3}\ Bh$. In the quadrangular pyramid composing the remainder of the frustum $CABD = \frac{1}{3}\ bh$. Then as to tetrahedron $CBDF$, drawing CG parallel to AD and GH

parallel to FE, and also the lines GB, GD, we obtain $CBED$ equivalent to $GBED$, or $BDGE$, from having the same base DBE, and the same height (for the vertices C and G are on a plain parallel to BE, and therefore to the plane DBE)[1].

Fig. 121.

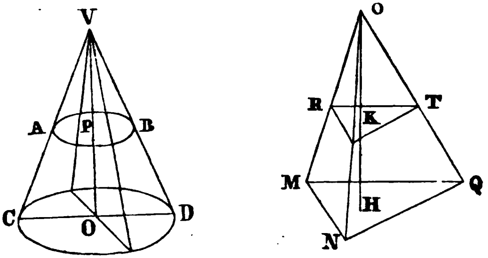
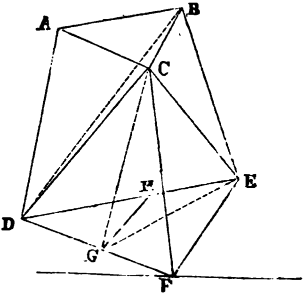


Fig. 122.



Then triangle DHG being equal to ABC, and triangles GED, GHD, having their bases ED and HD in a straight

[1] Memento du Baccalauréat ès Lettres, par M. Bos., op. cit., Théorème, No. 297, p. 198.

Manuel du Baccalauréat ès Lettres. Quatrième Partie. Eléments des Sciences. Trente-huitième Édition, revue et corrigée (1866-67). Théorème 113.

line, and their vertex common, and hence of the same height, we shall have:

$$b : EDG :: EH : ED.$$

Triangles EDG and EDF have the bases DG and DF in a straight line, and the same vertex ; therefore

$$EDG : B :: DG : DF.$$

But as the triangles DHG and EDF are similar:

$$EH : ED :: DG : DF.$$

then $b : EDG :: EDG : B.$

whence $EDG = \sqrt{Bb}.$ Therefore volume of Tetrahedron $DBGE = \frac{1}{3} h \sqrt{Bb}.$

Accordingly the volume of the truncated pyramid composed of tetrahedra, $CDEF$, $DABC$, and $CDEB$, (or its equivalent $BDGE$) is

$$V = \frac{1}{3} h B + \frac{1}{3} h b + \frac{1}{3} h \sqrt{B b}.$$

$$\text{or } V = \frac{1}{3} h (B + b + \sqrt{B b}.$$ [2]

Now the volume of a truncated cone of parallel bases must also be equal to one-third of its height, multiplied by the sum of its bases, and their mean proportional.

For the measure of the cone CVD = measure of the pyramid MOQ (Fig. 121.) Cutting the cone by a plane AB (Fig. 121) parallel to its base, we have:

$$\text{Volume of truncated cone} = \frac{1}{3} \pi H (R^2 + r^2 + R r),$$

where R and r are radii of its two bases, and $R\ r$ a mean proportional between them. The student can apply the Formulæ to Figures 121 as an exercise [3].

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[2] Amiot, Eléments, op. cit., p. 254.

[3] Amiot, Leçons 17 et 18, Théorème vi. Figures dans l'Espace.

THE END.

NOTES.

NOTE A. TO SECTION XXII. No. 99.

Calculation of the Ratio of the Circumference to Diameter, by the Method of Series employed by Candidates for admission into the Polytechnic School (Paris).

The calculation of π occupies an important place, in the programme for admission into the Polytechnic School at Paris; but no work exists presenting it exactly as it is required. It has therefore appeared desirable to add it in a Note to this volume, and to subjoin the calculation with logarithms, as prescribed in the same programme.

We have simplified this calculation as far as possible, so as to render the work less arduous, and we have given at the end some numerical examples in connexion with the question of relative errors.

Development of arc tang. x:—

We shall first consider the decreasing geometrical progression: $1 : x^4 : x^8 : x^{12} : x^{4(n-1)}$, in which x designates necessarily a quantity less than unity. Taking its sum, we obtain

$$1 + x^4 + x^8 + x^{12} + \dots x^{4(n-1)} = \frac{1 - x^{4n}}{1 - x}$$

$$= \frac{1}{1 - x^4} - \frac{x^{4n}}{1 - x^4}$$

and multiplying both members by $1 - x^2$

$$1 - x^2 + x^4 - x^6 + x^8 - x^{10} \dots - x^{4n-2}$$

$$= \frac{1}{1 + x^2} - \frac{x^{4n}}{1 + x^2} \text{ which gives}$$

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + x^8 \dots - x^{4n-2} + \frac{x^{4n}}{1 + x^2}$$

The first member is the derived function[4] of the arc tang. x; let us represent this arc by $f(x)$, and then

$$f(x) = 1 - x^2 + x^4 - x^6 + \ldots - x^{4n-2} + \frac{x^{4n}}{1 + x^{4n}}$$

Let us suppose, moreover,

$$\phi(x) = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \frac{x^{4n-1}}{4n-1},$$

and let us take the derived functions of both members; we shall also find $\phi' x = 1 - x^2 + x^4 - x^6 + \ldots - x^{4n-2}$; so that we have

$$f'(x) = \phi'(x) + \frac{x^{4n}}{1 + x^2}, \text{ or } f'(x) - \phi'(x) = \frac{x^{4n}}{1 + x^2} \quad (1)$$

This being the case, let us represent by u a constant, positive quantity, less than unity. If we make x vary from o to u, the second member of the preceding equality (1) will always remain positive. Its denominator will be constantly greater than 1, and the fraction $\frac{x^{4n}}{1 + x^2}$ will be always less than x^{4n}; we shall therefore have for all these values of x:—

$$f'(x) - \phi'(x) > o, f'(x) - \phi'(x) < x^{4n},$$

or,

$$f'(x) - \phi'(x) > o, f'(x) - \phi'(x) - x^{4n} < o \quad (2)$$

The first members of these inequalities are the derived functions of the functions

$$f(x) - \phi(x), f(x) - \phi(x) - \frac{x^{4n+1}}{4n+1},$$

which, as they must have no value when $x = o$, cannot contain an arbitrary constant. As of the derived functions (2) the first is constantly positive, and the second constantly negative, the first of these functions is increasing and the second decreasing, in the interval of $x = o$, to $x = u$, and as they are nothing when $x = o$, the first is positive and the second negative, when $x = u$. This gives

$$f(u) - \phi(u) > o, f(u) - \phi(u) - \frac{u^{4n+1}}{4n+1}, < o$$

or,

$$f(u) > \phi(u), f(u) < \phi(u) + \frac{u^{4n+1}}{4n+1},$$

---

[4] The function $f(x)$ is called by the French mathematicians the *derived function* (fonction deriveé or simply deriveé) and it is the same as what is generally called the differential co-efficient by the writers on the differential calculus. Note, p. 194, Dr. Thomson's Algebra. London, 1854.

consequently $f(u)$ is comprised between $\phi(u)$ and $\phi(u)$ increased by $\frac{u^{4n+1}}{4n+1}$; it is therefore equal to $\phi(u)$, increased by a quantity less than $\frac{u^{4n+1}}{4n+1}$, which can be represented by $\frac{\theta u^{4n+1}}{4n+1}$ θ designating a fraction, comprised between o and 1. We have therefore: $f(u) = \phi(u) + \frac{\theta u^{4n+1}}{4n+1}$,

or arc tang. $u = \frac{u}{1} - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \ldots - \frac{u^{4n-1}}{4n-1} + \frac{\theta u^{4n+1}}{4n+1}$.

The quantity u being less than 1, u^{4n+1}, and therefore $\frac{\theta u^{4n+1}}{4n+1}$, tend towards o, in proportion as n increases:—Therefore we have the formula, replacing u by x,

$$\text{arc tang. } x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots \quad (3)$$

therefore the second member is converging.

In this formula, let us substitute $\frac{1}{y}$ for x, y being superior to unity; we then have

$$\text{arc tang. } \frac{1}{y} = \frac{1}{y} - \frac{1}{3y^3} + \frac{1}{5y^5} - \frac{1}{7y^7} + \ldots\ldots$$

Now, it is known that arc tang. $\frac{1}{y}$ = arc cot. $y = \frac{\pi}{2}$ − arc tang. y; then by substitution and transposition

$$\text{arc tang. } y = \frac{\pi}{2} - \frac{1}{y} + \frac{1}{3y^3} - \frac{1}{5y^5} + \frac{1}{7y^7} - \ldots \quad (4)$$

for the values of y greater than unity.

Formulæ helping in calculating π:—

Formulæ (3) and (4) remain converging, however small the difference δ may be between unity and the tangent. Therefore they are still exact when δ is infinitely small, or when $x = 1$, $y = 1$; in this case they give

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

This formula we owe to Leibnitz. It is remarkable for the great simplicity of the fractions composing the series; it has little convergence, even when the subtractions indicated are effected, which give

$$\frac{\pi}{8} = \frac{1}{1\cdot3} + \frac{1}{5\cdot7} + \frac{1}{9\cdot11} + \ldots$$

A more converging series is obtained by assuming $x = \frac{1}{\sqrt{3}}$ which is the half side of the regular circumscribed hexagon. In this case we have arc tang. $x = \frac{\pi}{6}$ so that

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \ldots\right).$$

The most advantageous method to follow in calculating π is to decompose the arc $\frac{\pi}{4}$ into arcs, of which the tangents may be sufficiently small.

1st. Making $\frac{\pi}{4} = a + b$, which giv

$$tg\,(a+b) = 1 = \frac{tg\,a + tg\,b}{1 - tg\,a\,tg\,b} \text{ and } tg\,b = \frac{1 - tg\,a}{1 + tg\,a}$$

It may be seen at once that it will be sufficient to determine for $tg\,a$ and $tg\,b$ two simple and rational fractions, satisfying this condition.

For $tg\,a = \frac{1}{2}$, you find $tg\,b = \frac{1}{3}$, and with the help of formula (3),

$$a = \frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \ldots$$

$$b = \frac{1}{3} - \frac{1}{3.3^3} + \frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7} + \ldots$$

from which, by adding, we derive

$$a + b = \frac{\pi}{4} = \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) - \frac{1}{7}\left(\frac{1}{2^7} + \frac{1}{3^7}\right) +$$

$$= \frac{5}{6} - \frac{1}{3} \cdot \frac{35}{6^3} + \frac{1}{5} \cdot \frac{275}{6^5} - \frac{1}{7} \cdot \frac{2435}{6^7} + \ldots.$$

2nd. If we make $\frac{\pi}{4} = 2a + b$ we shall have $tg\,(2\,a + b) = 1$

$$= \frac{tg\,2\,a + tg\,b}{1 - tg\,2\,a\,tg\,b} = \frac{2\,tg\,a - tg^2\,a\,tg\,b + tg\,b}{1 - tg^2\,a - 2\,tg\,a\;tg\,b}$$

$$\text{and } tg\,b = \frac{1 - 2\,tg\,a - tg^2\,a}{1 + 2\,tg\,a - tg^2\,a}.$$

For $tg\,a = \frac{1}{3}$, gives $tg\,b = \frac{1}{7}$; consequently: $2a = \frac{2}{3} = \frac{2}{3 \cdot 3^3} + \frac{2}{5 \cdot 3^5} - \frac{2}{7 \cdot 3^7} + \ldots.$

$$b = \frac{1}{7} - \frac{1}{3 \cdot 7^3} + \frac{1}{5 \cdot 7^5} + \frac{1}{7 \cdot 7^7} + \cdots$$

and $$\frac{\pi}{4} = \left(\frac{2}{3} + \frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3} + \frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5} + \frac{1}{7^5}\right) - \frac{1}{7}\left(\frac{2}{3^7} + \frac{1}{7^7}\right) + \ldots$$

3rd. If we had made $\frac{\pi}{4} = 2a - b$ we should have found

$$tg\,(2a - b) = 1 = \frac{2\,tg\,a + tg^2 a\,tg\,b - tg\,b}{1 - tg^2 a + 2\,tg\,a\,tg\,b}$$

from which might have been derived

$$tg\,b = \frac{tg^2 a + 2\,tg\,a - 1}{1 + 2\,tg\,a - tg^2 a}$$

and the value $tg\,a = \frac{1}{2}$ would have given $tg\,b = \frac{1}{7}$. This would have given

$$\frac{\pi}{4} = \left(1 - \frac{1}{7}\right) - \frac{1}{3}\left(\frac{1}{2^2} - \frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5} + \frac{1}{7^5}\right) - \frac{1}{7}\left(\frac{1}{2^6} + \frac{1}{7^7}\right) + \cdots$$

4th. Making $\frac{\pi}{4} = a + b + c$,

we have $\frac{\pi}{4} - a = b + c$,

and taking the tangents,

$$\frac{1 - tg\,a}{1 + tg\,a} = \frac{tg\,b + tg\,c}{1 - tg\,b\,tg\,c};$$

this gives $$tg\,a = \frac{1 - tg\,b - tg\,c - tg\,b\,tg\,c}{1 + tg\,b + tg\,c - tg\,b\,tg\,c}.$$

When $tg\,b = \frac{1}{2}$, $tg\,c = \frac{1}{5}$, we find $tg\,a = \frac{1}{8}$

You thus obtain: $$\frac{\pi}{4} = \left(\frac{1}{2} + \frac{1}{5} + \frac{1}{8}\right) - \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{5^3} + \frac{1}{8^3}\right) + \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{5^5} + \frac{1}{8^5}\right) + \cdots$$

5th. The most rapid process is to make $\frac{\pi}{4} = 4a - b$.

It results from this, making $tg\,a = \alpha$, $tg\,b = \beta$.

$$g\,(4a - b) = \frac{4\alpha - 4\alpha^3 - (1 - 6\alpha^2 + \alpha^4)\,\beta}{1 - 6\alpha^2 + \alpha^4 + (4\alpha - 4\alpha^3)\,\beta}$$

$$tg\,b = \frac{a^4 + 4\,a^3 - 6\,a^2 - 4\,a + 1}{-\,a^4 + 4\,a^3 + 6\,a^2 - 4\,a - 1}$$

The English Geometrician Machin made $tg\,a = \frac{1}{5}$, and he obtained $tg\,b = \frac{1}{239}$: he was thus led to determine the value of π by help of the double series.

$$(5)\ \pi = 16\left(\frac{1}{5} - \frac{1}{3{\cdot}5^3} + \frac{1}{5{\cdot}5^5} - \frac{1}{7{\cdot}5^7} + \ldots\right)$$
$$- 4\left(\frac{1}{239} - \frac{1}{3{\cdot}239^3} + \frac{1}{5{\cdot}239^5} - \ldots\right)$$

where π takes the form of a difference $A - B$

Numerical calculation of π. Let us endeavour by help of formula (5) to calculate π with 15 exact decimals, that is, to less than $\frac{1}{10^{16}}$. It suffices, according to the rule to estimate A by defect and B by excess to less than $\frac{1}{2} \cdot \frac{A - B}{10^{16}}$

To determine a number inferior to π we shall direct attention to the fact that as the circle $\pi\,R^2$ includes the inscribed square $2\,R^2$, π is greater than 2; this gives as a consequence, $A - B > 2$ and $\frac{1}{2}. \frac{A - B}{10^{16}} > \frac{1}{10^{16}}$.

Therefore it is sufficient to calculate the two terms of the difference to less than one unit of the sixteenth decimal place. In the first series, A, the student will stop at the term $\frac{16}{23{\cdot}5^{23}}$

For the inquiry brings on the following term :—

$$\frac{16}{25{\cdot}5^{25}} < \frac{1}{5^{25}} = \frac{2^{25}}{10^{25}} < \frac{2^{13}2^{13}}{10^{25}};$$

Now $2^{13} = 8224 < 10^4$; consequently we get

$$\frac{16}{25{\cdot}5^{25}} < \frac{10^8}{10^{25}} = \frac{1}{10^{17}};$$

and as each of the terms coming afterwards is less than the 10th part of the preceding term, the sum total of the terms following $\frac{16}{23{\cdot}5^{23}}$ is less than the limit $\frac{1}{9{\cdot}10^{16}}$ of the sum of the terms of the progression $\frac{1}{10^{17}} + \frac{1}{10^{18}} + \frac{1}{10^{19}} + \ldots$ and consequently less than $\frac{1}{10^{16}}$.

In the second series, *B*, it is needless to go beyond term $\frac{4}{7 \cdot 239^7}$ as the following term $\frac{4}{9 \cdot 239^9}$ is less than $\frac{4}{8 \cdot 200^9} = \frac{1}{2^{10}.\ 10^{18}}$.

Accordingly the calculations to be effected relate to the first twelve terms of *A*, and the first four terms of *B*.

The first and third term of *A* can be exactly converted into decimals.

The other twelve terms will be calculated each to less than $\frac{1}{10^{17}}$: the total error will be less than $\frac{10}{10^{17}}$, or than $\frac{1}{10^{16}}$.

The same calculations will be made with the terms of *B*, each to less than $\frac{1}{10^{17}}$.

Note B to Section XLIV.

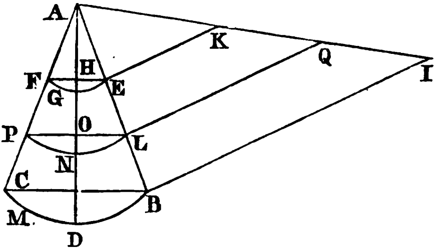
If we suppose a secant plane not passing through the summit of the cone, there are three cases to distinguish :—

1. The secant plane cuts all the generators of the cone, on the same side of the summit. It is then proved in higher geometry that the section is an ellipse.

2. The secant plane cuts the two folds of the cone; the curve therefore obtained is composed of two separate parts, open, and capable of indefinite prolongation; it has received the name of hyperbola, and is studied in higher geometry, or Conic Sections.

3. The secant plane only cuts one of the folds of the cone, and is parallel to a tangent plane; the section is then an unclosed curve, called *parabola*, also studied in more advanced mathematics [1].

Note C to No. 153 (notes 5 and 6).



[1] Géometrie Élémentaire, par H. Bos, Deuxième Anneé, p. 108 (No. 189).